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Discussion paper n. 45/2025

Salvatore Corrente (a), Salvatore Greco (a), Silvano Zappalà (a):

(a) University of Catania

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Keywords: Multiple Criteria Analysis; Deck of Cards-Based Ordinal Regression; Scaling procedures; Robust recommendations; Multiple Criteria Hierarchy Process.

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Deck of Cards method for Hierarchical, Robust and Stochastic Ordinal Regression

Salvatore Corrente[a], Salvatore Greco[a], Silvano Zappalà[a]

[a]*Department of Economics and Business, University of Catania, Corso Italia, 55, 95129 Catania, Italy*

Abstract: We consider the recently introduced application of the Deck of Cards Method (DCM) to ordinal regression proposing two extensions related to two main research trends in Multiple Criteria Decision Aiding, namely scaling and ordinal regression generalizations. On the one hand, procedures, different from DCM (e.g. AHP, BWM, MACBETH) to collect and elaborate Decision Maker's (DM's) preference information are considered to define an overall evaluation of reference alternatives. On the other hand, Robust Ordinal Regression and Stochastic Multicriteria Acceptability Analysis are used to offer the DM more detailed and realistic decision-support outcomes. More precisely, we take into account preference imprecision and indetermination through a set of admissible comprehensive evaluations of alternatives provided by the whole set of value functions compatible with DM's preference information rather than the univocal assessment obtained from a single value function. In addition, we also consider alternatives evaluated on a set of criteria hierarchically structured. The methodology we propose allows the DM to provide precise or imprecise information at different levels of the hierarchy of criteria. Like scaling procedures, the compatible value function we consider can be of a different nature, such as weighted sum, linear or general monotone value function, or Choquet integral. Consequently, the approach we propose is versatile and well-equipped to be adapted to DM's characteristics and requirements. The applicability of the proposed methodology is shown by a didactic example based on a large ongoing research project in which Italian regions are evaluated on criteria representing Circular Economy, Innovation-Driven Development and Smart Specialization Strategies.

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1. Introduction

Complex decisions require to take into consideration a plurality of points of view. For example, decisions related to circular economics (Stahel, 2016) have to be based on environmental aspects, economic aspects such as gross domestic product or employment rate, and also sustainability (Elliott, 2012) or smart specialization (McCann and Ortega-Argilés, 2015) aspects. Evaluation and comparison of different alternatives in complex decision problems – in our example, feasible economic policies and strategies – ask for adequately articulated models embedded in advanced decision support procedures. They allow to aggregate partial evaluations with respect to the many considered criteria into a global evaluation. In addition, observe that in such complex problems, many heterogeneous stakeholders, as well as several experts in different domains,

Email addresses: salvatore.corrente@unict.it (Salvatore Corrente), salgreco@unict.it (Salvatore Greco), silvano.zappala@phd.unict.it (Silvano Zappalà)

are involved together with a plurality of policy-makers. The results supplied by the formal decision model adopted for these types of complex problems heavily depend on the adopted parameters, so it is necessary to verify the stability of the obtained recommendation at the variation of the considered parameters. Just to give an example, if the trade-offs between the different criteria are represented by the weights assigned to them, it is reasonable to check if and how the comparisons between the alternatives change with a variation of these weights. Moreover, all the above-mentioned actors in the decision process – experts, stakeholders and policy-makers – in general, have no expertise in the methodology and techniques for decision analysis. Consequently, the decision support procedure for complex decisions needs:

- to permit handling several heterogeneous criteria;
- to collect the preference information from the different actors with a procedure as simple as possible;
- to collect rich preference information permitting to define in the most precise way the parameters of the decision models required to deal with the decision problem at hand;
- to provide information about the stability of the comparisons between alternatives when the parameters of the decision model are perturbed.

Multiple Criteria Decision Aiding (MCDA) (for an updated collection of state-of-the-art surveys see Greco et al. 2016, while for a review of the evolution of MCDA over the past 50 years, with a discussion on the perspectives and a future research agenda, see Greco et al. 2024) has developed a certain number of concepts, methods, techniques and procedures that can deal with the above requirements. Among them, we consider the ordinal regression (Jacquet-Lagreze and Siskos, 1982). It asks the Decision Maker (DM) to compare pairwise a subset of the considered alternatives, called reference alternatives, in terms of preference. These comparisons define a preference model, very often a value function, aiming to represent in the most faithful way the DM's preferences. As, in general, there is a plurality of value functions compatible with the DM's preferences, a theoretical development of the ordinal regression, the Robust Ordinal Regression (ROR) (Greco et al., 2008), proposes to considers the whole set of decision models compatible with DM's preferences answering to robustness concerns. With respect to the richness of the preference information provided by the DM, recently it has been proposed to extend the ordinal regression paradigm considering preference information related not only to the mere preference between reference alternatives but to its intensity too (Barbati et al., 2024). This means the DM can specify that, among three alternatives a, b, and c, not only a is preferred to b and b to c, but also that a is preferred to b more strongly than b to c. To enable the DM to present this type of information in a clear and understandable way, the Deck of Cards Method (DCM) (Figueira and Roy, 2002; Abastante et al., 2022) has been applied. To this aim, the DM is initially provided with a card for each reference alternative and a certain number of blank cards. Firstly, the DM rank orders the cards according to their preferences on the related alternatives. Secondly, the DM expresses the intensity of these preferences including a certain number of blank cards between two consecutive alternatives so that the greater the number of blank cards, the greater the preference between the alternatives. Once this preference information has been provided by the DM and a value function (for example, a weighted sum, an additive value function or a Choquet integral) has been selected to restore it, the instance of the value function that better represents these preferences can be obtained solving a linear programming problem. In particular, it is solved by minimizing the sum of the deviations between the values given to the reference alternatives by the DCM and the value assigned by the induced value function that is then used to give a value to all considered alternatives (not only reference ones). This approach has been called Deck-of-Cards-based Ordinal Regression (DOR) (Barbati et al., 2024). It has the advantage of collecting very rich preference information, that is, not only information about the preference ranking of reference alternatives as in ordinal regression but also information related to the intensity of preference between them. Even if DOR considers richer information than ordinal regression, it keeps its main weak points. In particular, DOR does not consider the possibility that multiple value functions may represent the DM's preferences with comparable accuracy. Moreover, DOR does not admit that the DM could provide imprecise preference information, for example, because they cannot exactly say how many blank cards have to be included between two alternatives. Another aspect not considered in DOR is that the set of criteria may be structured in a hierarchical way. Consequently, in this paper, we propose some extensions of DOR permitting to handle all the above problems and, more precisely:

- to define overall evaluations of reference alternatives by procedures collecting and elaborating DM's preference information different from DCM, such as AHP (Saaty, 1977), BWM (Rezaei, 2015) and MACBETH (Bana e Costa and Vansnick, 1994), which, for different reasons (for example because already known by the DM or the analyst) can be more appropriate in the specific decision problem at hand;

- to consider the whole set of value functions compatible with the DM's preferences for DOR method, applying the ROR (Greco et al., 2008). It builds a necessary and a possible preference relation holding in case an alternative is preferred to another for all or for at least one compatible value function;

- to consider the whole set of value functions compatible with the DM's preferences for DOR method, applying the Stochastic Multicriteria Acceptability Analysis (SMAA) (Lahdelma et al., 1998; Pelissari et al., 2020). Based on a sample of compatible value functions, it computes the frequency with which an alternative is ranked in a certain position or the frequency with which an alternative is preferred to another one;

- to generalize the DOR method by taking into account the possibility that the DM provides imprecise information about the number of blank cards between reference alternatives;

- to permit the DOR approach to take into account problems presenting a set of criteria hierarchically structured. This allows the DM to give preference information and to obtain final recommendations on the considered problem both at global and partial levels. To this aim, the Multiple Criteria Hierarchy Process (MCHP) (Corrente et al., 2012) will be adapted to this context.

The proposed extensions make DOR method very flexible and adaptable to real-world applications, taking into account two main research trends in MCDA: on the one hand, the scaling procedures such AHP, BWM and MACBETH, and, on the other hand, the extensions of ordinal regression, that is, ROR and Stochastic Ordinal Regression (Kadziński and Tervonen, 2013).

The paper is structured in the following way. In the next section, we introduce the basic framework recalling the basic principles of DOR methodology. Section 2 extends DOR to take into account the previously mentioned issues discussing in detail the possibility of applying scaling procedures different from the DCM. The DOR's extensions to ROR, SMAA and MCHP are provided in Sections 3.1, 3.2 and 3.3, respectively. How to include imprecise information provided by the DM in DOR is described in Section 4. A didactic example, based on a large ongoing research project, is presented in Section 5 to illustrate the application of the new proposal. Finally, Section 6 collects conclusions and future directions of research.

2. The Deck of Cards based Ordinal Regression

Let us denote by $A = \{a_1, a_2, \ldots\}$ a set of alternatives evaluated on a coherent family of criteria (Roy, 1996) $G = \{g_1, \ldots, g_n\}$. We shall assume that each criterion $g_i \in G$ is a real-valued function $g_i : A \rightarrow \mathbb{R}$ and, consequently, $g_i(a_j) \in \mathbb{R}$ is the evaluation of $a_j \in A$ on criterion g_i. For each $g_i \in G$, $X_i = \{x_i^1, \ldots, x_i^{m_i}\}$ is the whole set of evaluations that can be taken on g_i and they are such that $x_i^1 < \cdots < x_i^{m_i}$. In the following, for the sake of simplicity and without loss of generality, let us assume that all criteria have an increasing direction of preference (the greater $g_i(a_j)$, the better a_j is on g_i). Thus, $(x_1^{m_1}, \ldots, x_n^{m_n})$ represents the ideal alternative, achieving the highest possible evaluation across all criteria, whereas $(x_1^1, \ldots, x_n^1)$ represents the anti-ideal alternative, with the lowest possible evaluation on all criteria.

Looking at the evaluations of the alternatives on the considered criteria, the only objective information that can be obtained is the dominance relation for which $a_{j_1} \in A$ dominates $a_{j_2} \in A$ iff a_{j_1} is at least as good as a_{j_2} for all criteria and better for at least one of them ($g_i(a_{j_1}) \geqslant g_i(a_{j_2})$, for all $i = 1, \ldots, n$, and there exists at least one $g_i \in G$ such that $g_i(a_{j_1}) > g_i(a_{j_2})$). Even if it is objective that if a_{j_1} dominates a_{j_2}, then a_{j_1} is at least as good as a_{j_2}, the dominance relation is quite poor since, comparing two alternatives, in general, one is better than the other for some criteria but worse for the others. For this reason, to provide a final recommendation on the alternatives at hand, one needs to aggregate the evaluations taken by any alternative $a \in A$ on the criteria from G. In the field of MCDA, three main approaches have been proposed for aggregation of criteria:

1. using a value function $U : \mathbb{R}^n \rightarrow \mathbb{R}$ non decreasing in each of its arguments, such that, for any $a_{j_1}, a_{j_2} \in A$, if $U(g_1(a_{j_1}), \ldots, g_n(a_{j_1})) \geqslant U(g_1(a_{j_2}), \ldots, g_n(a_{j_2}))$, then a_{j_1} is comprehensively at least as good as a_{j_2} (Keeney and Raiffa, 1976);
2. using an outranking relation S on A defined on the basis of evaluations taken by alternatives from A on criteria from G, such that for any $a_{j_1}, a_{j_2} \in A, a_{j_1} S a_{j_2}$ means a_{j_1} is comprehensively at least as good as a_{j_2}; the relation S is reflexive, but, in general, neither transitive nor complete (Roy, 1996);
3. using a set of decision rules expressed in natural language, such as, for example, if "a is fairly preferred to a' on criterion g_{i_1} and extremely preferred on criterion g_{i_2}, then a is comprehensively at least as good as a'"; the decision rules are induced from some examples of decisions provided by the DM (Greco et al., 2001).

In this paper, we consider the aggregation of criteria through a value function U and we take into consideration a recently proposed methodology called DOR (Deck of Cards based Ordinal Regression) (Barbati et al., 2024). It collects the preference information from the DM using the Deck of Cards method (DCM) (Figueira and Roy, 2002; Abastante et al., 2022) and defines the parameters of the value function U using an ordinal regression approach (Jacquet-Lagrèze, 1982) by means of a mathematical programming problem that allows to represent the DM's preferences.

Considering a set of reference alternatives $A^R \subseteq A$, in the following, we shall briefly review the main steps of the DOR method:

1. The DM has to rank-order alternatives in A^R from the least to the most preferred in sets $L_1, L_2, \ldots, L_s \subseteq A^R$. Alternatives in L_{h+1} are preferred to alternatives in L_h for all $h = 1, \ldots, s-1$ and alternatives in L_h are indifferent among them for all $h = 1, \ldots, s$;
2. The DM can include a certain number of blank cards e_h between sets L_h and L_{h+1}. The greater e_h, the more alternatives in L_{h+1} are preferred to alternatives in L_h. Observe that the absence of blank cards

between L_h and L_{h+1} does not mean that the alternatives in L_h are indifferent to the alternatives in L_{h+1} but that the difference between the value of the alternatives in L_{h+1} and the value of the alternatives in L_h is minimal;

3. Following Abastante et al. (2022), the DM has to provide the number of blank cards e_0 between the "fictitious zero alternative" and the set of the least preferred alternatives, that is, L_1. In this context, the "fictitious zero alternative" is a fictitious alternative a_0 having a null value, that is, $U(a_0) = 0$. To make coherent the notation introduced above, let us assume $L_0 = \{a_0\}$;
4. Each alternative $a \in A^R$ is assigned a value $\nu(a)$ such that if $a \in L_{h+1}$ and $a' \in L_h$, $\nu(a) = \nu(a') + (e_h + 1)$, $h = 1, \ldots, s-1$ and if $a \in L_1$, then $\nu(a) = e_0 + 1$. Consequently, in general, for $a \in L_h$ we have: $\nu(a) = \sum_{p=0}^{h-1}(e_p + 1)$;
5. The parameters of the value function U are determined in a way that for all $a \in A^R$, $U(a)$ deviates as less as possible from $k \cdot \nu(a)$ with k a scalarizing constant that can be interpreted as the value of a single blank card. More precisely, for all $a \in A^R$, one considers the positive and negative deviations $\sigma^+(a)$ and $\sigma^-(a)$ between $U(a) = U(g_1(a), \ldots, g_n(a))$ and $k \cdot \nu(a)$, that is $U(a) - \sigma^+(a) + \sigma^-(a) = k \cdot \nu(a)$. The sum of all deviations $\sigma^+(a)$ and $\sigma^-(a), a \in A^R$, is then minimized solving the following optimization problem having as unknown variables the constant k and the deviations $\sigma^+(a)$ and $\sigma^-(a)$, in addition to the parameters of the value function U

$$\overline{\sigma} = \min \sum_{a \in A^R} (\sigma^+(a) + \sigma^-(a)), \text{ subject to,}$$
$$\left.\begin{array}{l} E^{Model} \\ U(a) \geqslant U(a'), \text{ for all } a, a' \in A^R \text{ such that } \nu(a) \geqslant \nu(a'), \\ U(a) - \sigma^+(a) + \sigma^-(a) = k \cdot \nu(a) \text{ for all } a \in A^R, \\ k \geqslant 0, \\ \sigma^+(a) \geqslant 0, \ \sigma^-(a) \geqslant 0 \text{ for all } a \in A^R, \end{array}\right\} E^{DM} \tag{1}$$

where:

- E^{Model} is the set of technical constraints related to the considered function U: for example, if U is formulated as a *weighted sum*, we have:

$$\left.\begin{array}{l} U(a) = \sum_{i=1}^{n} w_i \cdot g_i(a) \text{ for all } a \in A, \\ w_i \geqslant 0, \text{ for all } i = 1, \ldots, n, \\ \sum_{i=1}^{n} w_i = 1, \end{array}\right\} E^{Model}_{WS}$$

while, if U is formulated as a *general additive value function*, we have:

$$\left.\begin{array}{l} U(a)=\sum_{i=1}^{n} u_i(g_i(a)) \text{ for all } a\in A,\\ u_i\left(x_i^{f-1}\right)\leqslant u_i\left(x_i^{f}\right), \text{ for all } i=1,\ldots,n \text{ and for all } f=2,\ldots,m_i,\\ u_i\left(x_i^{1}\right)=0, \text{ for all } i=1,\ldots,n,\\ \sum_{i=1}^{n} u_i\left(x_i^{m_i}\right)=1. \end{array}\right\} E_{GA}^{Model}$$

More details on different formulation of value function U and, consequently, of the set of constraints E^{Model} will be given in the Appendix A;

- $U(a)\geqslant U(a')$ for all $a,a'\in A^R$ such that $\nu(a)\geqslant\nu(a')$, imposes that the ranking of alternatives from A^R provided by the value function U is concordant with the one provided by the DM,
- $U(a)-\sigma^+(a)+\sigma^-(a)=k\cdot\nu(a)$ imposes that the value of $a\in A^R$ is proportional to $\nu(a)$,
- $k\geqslant 0$ imposes the non-negativity of the scalarizing constant k,
- $\sigma^+(a)\geqslant 0,\ \sigma^-(a)\geqslant 0, a\in A^R$ impose the non-negativity of the positive and negative deviations.

The following cases can occur:

1. $\overline{\sigma}=0$ and $k>0$: in this case, the function U obtained as solution of the LP problem (1) represents the preferences of the DM without any error;

2. $\overline{\sigma}=0$ and $k=0$: in this case, the function U obtained as solution of the LP problem (1) does not represent the preferences of the DM since the scalarizing constant k is zero and, therefore, all alternatives in A have a null evaluation being indifferent among them;

3. $\overline{\sigma}>0$ and $k>0$: in this case, the function is able to represent the preferences of the DM with the minimal error $\overline{\sigma}$;

4. $\overline{\sigma}>0$ and $k=0$: in this case, the function U obtained as solution of the LP problem (1) does not represent the preferences of the DM since, as in above point 2., the scalarizing constant k is null.

In the following, we shall present a small didactic example showing the DOR application.

Example 2.1. *Consider four regions* $\mathtt{R}_1,\mathtt{R}_2,\mathtt{R}_3,\mathtt{R}_4$ *evaluated on a* $0-100$ *scale with respect to the three criteria of Circular Economy* (g_1)*, Innovation-Driven Development* (g_2)*, and Smart Specialization* (g_3)*, as shown in Table 1. Using the DCM and taking into consideration a "zero Region"* $\mathtt{R}_0$ *as a reference of a null*

Table 1: Evaluations of regions with respect to considered criteria

Regions	Circular Economy: g_1	Innovation-Driven Development: g_2	Smart Specialization: g_3
$\mathtt{R}_1$	90	100	80
$\mathtt{R}_2$	100	70	40
$\mathtt{R}_3$	30	50	60
$\mathtt{R}_4$	20	40	40

value level, the four regions $\mathtt{R}_1,\mathtt{R}_2,\mathtt{R}_3$ *and* $\mathtt{R}_4$ *are ordered from worst to best so that* $L_0=\{\mathtt{R}_0\}$, $L_1=\{\mathtt{R}_4\}$,

$L_2 = \{\mathtt{R_3}\}$, $L_3 = \{\mathtt{R_2}\}$ and $L_4 = \{\mathtt{R_1}\}$. Moreover, let e_s be the number of blank cards (written between square brackets [ ]) between L_s and L_{s+1}, $s = 0, \ldots, 3$, so that the complete preference information is the following:

$$\mathtt{R_0}\ [5]\ \mathtt{R_4}\ [2]\ \mathtt{R_3}\ [5]\ \mathtt{R_2}\ [2]\ \mathtt{R_1}.$$

By applying the DCM, we assign the following value to each project:

- $\nu(\mathtt{R_0} = [0, 0, 0]) = 0,$
- $\nu(\mathtt{R_4} = [20, 40, 40]) = \nu(\mathtt{R_0}) + e_0 + 1 = 6,$
- $\nu(\mathtt{R_3} = [30, 50, 60]) = \nu(\mathtt{R_4}) + e_1 + 1 = 9,$
- $\nu(\mathtt{R_2} = [100, 70, 40]) = \nu(\mathtt{R_3}) + e_2 + 1 = 15,$
- $\nu(\mathtt{R_1} = [90, 100, 80]) = \nu(\mathtt{R_2}) + e_3 + 1 = 18.$

Considering a value function $U(\cdot)$ expressed in terms of a weighted sum, the following LP problem has to be solved for the unknown variables $w_1, w_2, w_3, \sigma^+(\mathtt{R_i}), \sigma^-(\mathtt{R_i}), i = 1, \ldots, 4$, and k:

$$\begin{array}{l}
\overline{\sigma} = \min \sum_{i=1}^{4} \left(\sigma^+(\mathtt{R_i}) + \sigma^-(\mathtt{R_i})\right), \textit{ subject to,} \\
\left.\left.\begin{array}{l}
U(\mathtt{R_i}) = w_1 \cdot g_1(\mathtt{R_i}) + w_2 \cdot g_2(\mathtt{R_i}) + w_3 \cdot g_3(\mathtt{R_i}) \;\; i = 1, \ldots, 4, \\
w_1 \geqslant 0, w_2 \geqslant 0, w_3 \geqslant 0, \\
w_1 + w_2 + w_3 = 1,
\end{array}\right\} E_{WS}^{Model} \atop
\begin{array}{l}
U(\mathtt{R_1}) \geqslant U(\mathtt{R_2}) \geqslant U(\mathtt{R_3}) \geqslant U(\mathtt{R_4}), \\
U(\mathtt{R_i}) - \sigma^+(\mathtt{R_i}) + \sigma^-(\mathtt{R_i}) = k \cdot \nu(\mathtt{R_i}), \;\; i = 1, \ldots, 4, \\
k \geqslant 0, \\
\sigma^+(\mathtt{R_i}) \geqslant 0, \sigma^-(\mathtt{R_i}) \geqslant 0, \;\; i = 1, \ldots, 4.
\end{array}\right\} E^{DM}
\end{array} \tag{2}$$

The solution of the LP problem (2) yields a sum of errors $\overline{\sigma} = 1.176$, a scaling constant $k = 4.902$ and the following weights for the considered criteria $w_1 = 0.471, w_2 = 0.176, w_3 = 0.353$. Using them, one can obtain the values listed in Table 2. ■

Table 2: Scores assigned to regions by the value function $U(\cdot)$ obtained solving the LP problem (2)

Regions	$U(\cdot)$	$\nu(\cdot)$	$k \cdot \nu(\cdot)$	$\sigma^+(\cdot)$	$\sigma^-(\cdot)$
$\mathtt{R_1}$	88.24	18	88.24	0	0
$\mathtt{R_2}$	73.53	15	73.53	0	0
$\mathtt{R_3}$	44.12	9	44.12	0	0
$\mathtt{R_4}$	30.59	6	29.21	1.18	0

As previously explained, k represents the value of a blank card and, consequently, it has to be greater than zero to avoid that all alternatives from A are indifferent between them. Therefore, in order to ensure that this happens, we solve in an iterative way the following LP problem:

$$\begin{array}{l}
k^* = \max k \text{ subject to,} \\
\left.\begin{array}{l}
E^{DM}, \\
\sum_{a \in A^R} \left(\sigma^+(a) + \sigma^-(a)\right) \leqslant \overline{\sigma} + \eta\left(\overline{\sigma}\right),
\end{array}\right\} E^{DM'}
\end{array} \tag{3}$$

where $\eta(\overline{\sigma})$ is an admitted deterioration error with respect to the optimal value $\overline{\sigma}$ obtained solving (1). If $k^* = 0$, to keep the deterioration error as low as possible, $\eta(\overline{\sigma})$ should be increased in increments of 0.01, starting from 0, until $k^* > 0$.

In the following, we shall provide a small didactic example to show the necessity of maximizing the k value.

Example 2.2. *Consider three alternatives evaluated on two criteria with increasing direction of preference, as shown in Table 3:*

Table 3: Alternatives' evaluations on two criteria

Alternative	g_1	g_2
a	0.3	0.7
b	0.4	0.6
c	0.8	1

Table 4: DM's preference information

Alternative	*Level*
a	$\nu(a) = 100$
	$e_1 = 29$
b	$\nu(b) = 70$
	$e_0 = 69$

Suppose the DM provides preference information for alternatives a and b as shown in Table 4. Specifically, a is preferred to b, putting 69 blank cards between b and the zero level and 29 blank cards between a and b. Assuming the DM's preference model follows a general additive value function, the LP problem (1) has to be solved having as unknown variables $\sigma^+(a)$, $\sigma^-(a)$, $\sigma^+(b)$, $\sigma^-(b)$, $u_1(0.3)$, $u_1(0.4)$, $u_1(0.8)$, $u_2(0.6)$, $u_2(0.7)$, $u_2(1)$ *and* k *(see the LP problem (B.1) in Appendix B for the extended formulation of the problem to be solved in this case). Solving this problem, one finds the solution shown in Table 5.*

Table 5: Marginal value functions obtained solving the LP problem (B.1)

$u_1(0.3)$	$u_1(0.4)$	$u_1(0.8)$	$u_2(0.6)$	$u_2(0.7)$	$u_2(1)$	k	$\overline{\sigma}$
0	0	0	0	0	1	0	0

Here, $k = 0$ *means that the blank card has null value. In order to maximize it, one has to solve the LP problem (3) (see the LP problem (B.2) in Appendix B for the extended formulation of the problem to be solved). Setting* $\eta(\overline{\sigma}) = 0$ *in problem (B.2), one gets the marginal values shown in Table 6:*

Table 6: Marginal value functions obtained solving the LP problem (B.2)

$u_1(0.3)$	$u_1(0.4)$	$u_1(0.8)$	$u_2(0.6)$	$u_2(0.7)$	$u_2(1)$	k	$\overline{\sigma}$
0	0.4118	0.4118	0	0.5882	0.5882	0.0059	0

This value function accurately reflects the DM's preferences with a positive value for the blank card. ■

Some remarks are now at order:

1. LP problem (1) assumes that overall evaluations $\nu(\cdot)$ and $U(\cdot)$ are expressed on a ratio scale, that is, $U(a)/U(b) = \nu(a)/\nu(b)$ for all $a, b \in A$. If instead evaluations $\nu(\cdot)$ and $U(\cdot)$ are to be expressed on an interval scale, that is, $[U(a) - U(b)]/[U(c) - U(d)] = [\nu(a) - \nu(b)]/[\nu(c) - \nu(d)]$, for all $a, b, c, d \in A$ (Stevens, 1946), LP problem (1) has to be properly reformulated. More precisely, remembering that the admissible transformations for a ratio scale are the multiplications by a positive constant, that is $U(\cdot) = k \cdot \nu(\cdot), k > 0$, while the admissible transformations for an interval scale are the positive affine transformations, that is $U(\cdot) = k \cdot \nu(\cdot) + h$, $k > 0, h \in \mathbb{R}$, we can conclude that to handle interval scales the regression model (1) has to be reformulated replacing the constraint

$$U(a) - \sigma^+(a) + \sigma^-(a) = k \cdot \nu(a) \text{ for all } a \in A^R, \tag{4}$$

with the constraint

$$U(a) - \sigma^{+}(a) + \sigma^{-}(a) = k \cdot \nu(a) + h \ \text{ for all } a \in A^R. \tag{5}$$

Observe that if $\nu(\cdot)$ and $U(\cdot)$ are expressed on an interval scale, the information related to the number of blank cards e_0 between the worst alternative and the zero alternative is redundant, because it is not related to any difference $\nu(a) - \nu(b)$, $a, b \in A^R$. For the sake of simplicity, if not explicitly mentioned, we refer to the case in which $\nu(\cdot)$ and $U(\cdot)$ are expressed on a ratio scale. Anyway, what we discuss with respect to the ratio scale can be straightforwardly extended to the interval scale.

In the following, we shall continue the Example 2.1 showing how the described procedure changes once $U(\cdot)$ and $\nu(\cdot)$ are expressed on an interval scale.

Example 2.1 (continuation). *Assuming $\nu(\cdot)$ and $U(\cdot)$ are expressed on an interval scale, the solution of the LP problem (2) in which constraints*

$$U(\mathtt{R_i}) - \sigma^{+}(\mathtt{R_i}) + \sigma^{-}(\mathtt{R_i}) = k \cdot \nu(\mathtt{R_i}), \ \ i = 1, \ldots, 4$$

are replaced by constraints

$$U(\mathtt{R_i}) - \sigma^{+}(\mathtt{R_i}) + \sigma^{-}(\mathtt{R_i}) = k \cdot \nu(\mathtt{R_i}) + h, \ \ i = 1, \ldots, 4$$

yields a null sum of errors $\overline{\sigma}$, $k = 4.722$, $h = 1.667$ and the following weights for the considered criteria: $w_1 = 0.5$, $w_2 = 0.08$, $w_3 = 0.42$. Using them, one can obtain the values listed in Table 7. ■

Table 7: Scores assigned to regions by the value function $U(\cdot)$ expressed on an interval scale and obtained solving the LP problem (2)

Regions	$U(\cdot)$	$\nu(\cdot)$	$k \cdot \nu(\cdot) + h$	$\sigma^{+}(\cdot)$	$\sigma^{-}(\cdot)$
$\mathtt{R_1}$	88.67	18	88.67	0	0
$\mathtt{R_2}$	72.50	15	72.50	0	0
$\mathtt{R_3}$	44.17	9	44.17	0	0
$\mathtt{R_4}$	30.00	6	30.00	0	0

2. Until now, we considered the DCM to assign an overall evaluation $\nu(a)$ to the reference alternatives $a \in A^R$. However, for several reasons, the Decision Maker (DM) may encounter difficulties in using the DCM. For instance, when a high number of blank cards is required to evaluate the alternatives, the method may be perceived as essentially asking the DM to directly assess their values. In some cases, this perception could make the approach seem unrealistic, contradicting the original purpose of the DCM, introduced by Simos (1990) to reduce the DM's cognitive burden by using a limited number of cards, thus making it easier to manipulate a ratio scale. However, as already noted in Barbati et al. (2024), depending on the specific problem, the DM's prior experience, and their individual predisposition, alternative procedures can be used to define overall evaluations $\nu(a)$. Among the simplest of these scaling methods, let us remember the direct rating and point allocations (Doyle et al., 1997), as well as SMART and its extensions (Edwards, 1977; Edwards and Barron, 1994). Anyway, the most well-known among the methods proposed in literature to assign a value to alternatives on the basis of preference information provided by the DM are AHP (Saaty, 1977), BWM (Rezaei, 2015) and MACBETH (Bana e Costa and Vansnick, 1994). All these methods are based on pairwise comparisons of alternatives from A^R. For example, AHP provides evaluations $\nu(a), a \in A^R$, on a ratio scale, because it maintains the ratio $\nu(a)/\nu(b) = \nu'(a)/\nu'(b)$ between the evaluations of any $a, b \in A$, taking into

consideration the DM's pairwise comparisons $c_{a,b}$ of reference alternatives $a, b \in A^R$ in the classical 1-9 Saaty scale. Assuming that the DM retains b not preferred to a, the points in the scale have the following interpretation:

1- a and b are equally preferable, denoted by $c_{a,b} = 1$, translated to $\nu(a) = \nu(b)$,

3- a is moderately preferred to b, denoted by $c_{a,b} = 3$, translated to $\nu(a) = 3 \cdot \nu(b)$,

5- a is strongly preferred to b, denoted by $c_{a,b} = 5$, translated to $\nu(a) = 5 \cdot \nu(b)$,

7- a is very strongly preferred to b, denoted by $c_{a,b} = 7$, translated to $\nu(a) = 7 \cdot \nu(b)$,

9- a is extremely preferred to b, denoted by $c_{a,b} = 9$, translated to $\nu(a) = 9 \cdot \nu(b)$.

Values 2, 4, 6 and 8 denote a hesitation between 1-3, 3-5, 5-7 and 7-9, respectively. AHP considers the matrix $C = [c_{a,b}]_{a,b \in A^R}$, with $c_{a,b} = \frac{1}{c_{b,a}}$ for all $a, b \in A^R$. In general, there is no set of evaluations $\nu(a), a \in A^R$, satisfying all equations $\nu(a) = c_{a,b} \cdot \nu(b)$, $a, b \in A^R$. AHP computes the values $\nu(a), a \in A^R$ as the components of the eigenvector $\hat{\nu} = [\hat{\nu}(a)]_{a \in A^R}$ associated to the maximum eigenvalue of C (λ_{max}) normalized so that $\sum_{a \in A^R} \hat{\nu}(a) = 1$. Formally we have

$$\begin{cases} C \cdot \hat{\nu} = \lambda_{max} \cdot \hat{\nu} \\ \sum_{a \in A} \hat{\nu}(a) = 1. \end{cases} \tag{6}$$

Procedures different from the eigenvector have been proposed to obtain the values $\nu(a), a \in A^R$, from the judgments in the matrix $C = [c_{a,b}]_{a,b \in A^R}$ and, among them, we remember the logarithmic least squares (Crawford and Williams, 1985), the least squares (Jensen, 1984), the weighted least squares (Chu et al., 1979; Blankmeyer, 1987), the logarithmic least absolute values (Cook and Kress, 1988) and the geometric least square (Islei and Lockett, 1988).

Some considerations related to decision psychology and to the great number of pairwise comparison judgments $c_{a,b}, a, b \in A^R$ required by AHP to fill the matrix C suggested to consider only pairwise comparisons with respect to the best and the worst alternatives originating the BWM (Rezaei, 2015). A reduced number of pairwise comparison judgments is taken into account also in two recently proposed procedures (Abastante et al., 2019; Corrente et al., 2024) that correct the DM's direct rating of alternatives through the overall evaluations provided by AHP or BWM to a limited number of reference alternatives.

Requiring preference information similar to AHP, MACBETH provides evaluations $\nu(a), a \in A^R$, on an interval scale. In particular, assuming that the DM retains b not preferred to a, MACBETH considers pairwise comparisons $c^M_{a,b}, a, b \in A^R$, on a 0-6 scale, having the following interpretation:

0_M- a and b are equally preferable, denoted as $c^M_{a,b} = 0$,

1_M- a is very weakly preferred to b, denoted as $c^M_{a,b} = 1$,

2_M- a is weakly preferred to b, denoted as $c^M_{a,b} = 2$,

3_M- a is moderately preferred to b, denoted as $c^M_{a,b} = 3$,

4_M- a is strongly preferred to b, denoted as $c^M_{a,b} = 4$,

5_M- a is very strongly preferred to b, denoted as $c^M_{a,b} = 5$,

6_M- a is extremely preferred to b, denoted as $c^M_{a,b} = 6$.

Below, we consider the following MACBETH-like LP problem to define the overall evaluations $\nu(a), a \in A^R$:

$$
\begin{aligned}
&\max \gamma, \text{ subject to,} \\
&\nu(a^*) = 100,\ \nu(a_*) = 0 && (7.1)\\
&\left.\begin{array}{l}[\nu(a) - \sigma^+(a) + \sigma^-(a)] - [\nu(b) - \sigma^+(b) + \sigma^-(b)] = 0 \text{ if } c^M_{a,b} = 0,\\ [\nu(a) - \sigma^+(a) + \sigma^-(a)] - [\nu(b) - \sigma^+(b) + \sigma^-(b)] = \delta_e, \text{ if } c^M_{a,b} = e,\ e = 1, \ldots, 6,\end{array}\right\} a, b \in A^R && (7.2)\\
&\delta_{e+1} - \delta_e \geqslant \gamma, e = 1, \ldots, 5, && (7.3)\\
&\sum_{a \in A^R} \left(\sigma^+(a) + \sigma^-(a)\right) \leqslant \overline{\sigma}, && (7.4)\\
&\sigma^+(a) \geqslant 0, \sigma^-(a) \geqslant 0. && (7.5)
\end{aligned}
$$

Constraints (7.1)–(7.5) form E_{Macb}.

The value of $\overline{\sigma}$ is obtained solving the following auxiliary optimization problem

$$\overline{\sigma} = \max \sum_{a \in A^R} \left(\sigma^+(a) + \sigma^-(a)\right), \quad \text{subject to } E'_{Macb} \tag{8}$$

where E'_{Macb} is obtained by E_{Macb} replacing constraint (7.4) with $\gamma \geqslant \varepsilon$; ε is a small positive number defined to ensure that γ assumes a strictly positive value; a_* and a^* are the worst and the best alternatives in A^R, respectively.

Example 2.1 (continuation). *Applying AHP, let us assume the pairwise comparisons* $c_{\mathtt{R_i},\mathtt{R_j}}, i, j = 1, 2, 3, 4$, *shown in the following matrix:*

$$C = \begin{array}{c} \\ \mathtt{R_1} \\ \mathtt{R_2} \\ \mathtt{R_3} \\ \mathtt{R_4} \end{array} \begin{array}{c} \begin{array}{cccc} \mathtt{R_1} & \mathtt{R_2} & \mathtt{R_3} & \mathtt{R_4} \end{array} \\ \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 2 \\ 1/2 & 1/2 & 1 & 1 \\ 1/3 & 1/2 & 1 & 1 \end{pmatrix} \end{array}$$

Solving problem (6) one obtains the normalized eigenvector $\hat{\nu}$ *of* C *providing the following overall evaluations for the regions* $\mathtt{R_1}, \mathtt{R_2}, \mathtt{R_3}$ *and* $\mathtt{R_4}$: $\hat{\nu}(\mathtt{R_1}) = 0.36, \hat{\nu}(\mathtt{R_2}) = 0.33, \hat{\nu}(\mathtt{R_3}) = 0.16, \hat{\nu}(\mathtt{R_4}) = 0.15$. *As AHP expresses overall evaluations* $\nu(\mathtt{R_i}), i = 1, 2, 3, 4$, *on a ratio scale, replacing them in the LP problem (2) one gets the total error* $\overline{\sigma} = \sum_{i=1}^{4} \left(\sigma^+(\mathtt{R_i}) + \sigma^-(\mathtt{R_i})\right) = 8.25$ *and the results listed in Table 8 where* $k = 248.87$ *and the weights for the considered criteria are* $w_1 = 0.57, w_2 = 0.23, w_3 = 0.2$.

Table 8: Scores assigned to regions by the value function $U(\cdot)$ expressed on a ratio scale and obtained solving the LP problem (2) with respect to AHP overall evaluations $\nu(\mathtt{R_i}), i = 1, 2, 3, 4$.

Regions	$U(\cdot)$	$\nu(\cdot)$	$k \cdot \nu(\cdot)$	$\sigma^+(\cdot)$	$\sigma^-(\cdot)$
$\mathtt{R_1}$	90.31	0.36	90.31	0	0
$\mathtt{R_2}$	81.15	0.32	81.15	0	0
$\mathtt{R_3}$	40.58	0.16	40.58	0	0
$\mathtt{R_4}$	28.59	0.15	36.83	0	8.25

Considering MACBETH and assuming that the pairwise comparisons $c^M_{\mathtt{R_i},\mathtt{R_j}}, i, j = 1, 2, 3, 4$ *are collected* *as follows*

$$C^M = \begin{array}{c} \\ \mathtt{R_1} \\ \mathtt{R_2} \\ \mathtt{R_3} \\ \mathtt{R_4} \end{array} \begin{array}{c} \begin{array}{cccc} \mathtt{R_1} & \mathtt{R_2} & \mathtt{R_3} & \mathtt{R_4} \end{array} \\ \left(\begin{array}{cccc} 0 & 1 & 4 & 5 \\ & 0 & 3 & 4 \\ & & 0 & 1 \\ & & & 1 \end{array} \right) \end{array}$$

solving sequentially problems (8) and (7) one obtains the following overall evaluations for regions $\mathtt{R_1}, \mathtt{R_2}, \mathtt{R_3}$ *and* $\mathtt{R_4}$*:* $\nu(\mathtt{R_1}) = 100, \nu(\mathtt{R_2}) = 80, \nu(\mathtt{R_3}) = 20, \nu(\mathtt{R_4}) = 0$*. As MACBETH expresses overall evaluations* $\nu(\mathtt{R_i}), i = 1, 2, 3, 4$*, on an interval scale, using them in the LP problem (2) where eqs. (4) are replaced by eqs. (5) one gets a null total error* $\left(\sum_{i=1}^{4} \left(\sigma^+(\mathtt{R_i}) + \sigma^-(\mathtt{R_i}) \right) = 0 \right)$ *and the results listed in Table 9 where* $k = 0.607$*,* $h = 30$*, and the weights for the considered criteria are* $w_1 = 0.5, w_2 = 0.29, w_3 = 0.21$*.* ■

Table 9: Scores assigned to regions by the value function $U(\cdot)$ obtained solving the reformulation of LP problem (2) for interval scales with respect to MACBETH overall evaluations $\nu(\mathtt{R_i}), i = 1, 2, 3, 4$

Regions	$U(\cdot)$	$\nu(\cdot)$	$k \cdot \nu(\cdot) + h$	$\sigma^+(\cdot)$	$\sigma^-(\cdot)$
$\mathtt{R_1}$	90.71	100	90.71	0	0
$\mathtt{R_2}$	78.57	80	78.57	0	0
$\mathtt{R_3}$	42.14	20	42.14	0	0
$\mathtt{R_4}$	30	0	30.00	0	0

3. The number of cards $e_h, h = 0, \ldots, s-1$ of the DCM can be interpreted in a more ordinal form in the sense that instead of $U(a) - U(b) = k \cdot e_h$ for all $a \in L_{h+1}, b \in L_h$, we could consider the much weaker relation for which if $e_h \geqslant e_{h'}, h, h' = 0, \ldots, s$, then $U(a) - U(b) \geqslant U(c) - U(d)$, for all $a \in L_{h+1}$, $b \in L_h$, $c \in L_{h'+1}$ and $d \in L_{h'}$. Accepting this meaning of the blank cards, the regression problem (1) can be reformulated as follows:

$$\max \gamma \text{ subject to,}$$

$$\left.\begin{array}{ll} E^{Model} & (9.1) \\ U(a) \geqslant U(a') \text{ for all } a \in L_{h+1}, a' \in L_h, h = 0, \ldots, s-1, & (9.2) \\ U'(a) = U(a) - \sigma^+(a) + \sigma^-(a) \ \text{ for all } a \in A^R, & (9.3) \\ U'(a) - U'(b) = \delta_h \ \text{ for all } a \in L_{h+1} \text{ and } b \in L_h, h = 0, \ldots, s-1, & (9.4) \\ \delta_h - \delta_{h'} \geqslant \gamma \text{ if } e_h > e_{h'}, \ h, h' = 0, \ldots, s-1, & (9.5) \\ \sum_{a \in A^R} (\sigma^+(a) + \sigma^-(a)) \leqslant \overline{\sigma} & (9.6) \\ \sigma^+(a) \geqslant 0, \ \sigma^-(a) \geqslant 0 \text{ for all } a \in A^R. & (9.7) \end{array}\right\} E_{Ordinal}$$

The value of $\overline{\sigma}$ is obtained solving the following auxiliary LP problem

$$\overline{\sigma} = \min \sum_{a \in A^R} (\sigma^+(a) + \sigma^-(a)), \ \text{ subject to } \ E'_{Ordinal} \tag{10}$$

where $E'_{Ordinal}$ is obtained from $E_{Ordinal}$ replacing constraints (9.5) and (9.6) with the following one

$$\delta_h \geqslant \delta_{h'} \text{ if } e_h \geqslant e_{h'}, \quad h, h' = 0, \ldots, s-1.$$

Observe that the DM's preference information could be collected also in terms of ratio instead of difference between overall evaluations of alternatives, that is, if $e_h \geqslant e_{h'}, h, h' = 0, \ldots, s$, then $U(a)/U(b) \geqslant U(c)/U(d)$, for all $a \in L_{h+1}$, $b \in L_h$, $c \in L_{h'+1}$, $d \in L_{h'}$. This interpretation of the meaning of blank cards requires the following reformulation of the regression problem:

$$\max \gamma, \text{ subject to,}$$

$$\left.\begin{array}{ll}
E^{Model} & (11.1)\\
U(a) \geqslant U(a'), \text{ for all } a \in L_{h+1}, a' \in L_h, h = 1, \ldots, s-1, & (11.2)\\
U'(a) = U(a) - \sigma^+(a) + \sigma^-(a) \text{ for all } a \in A^R, & (11.3)\\
U'(a)/U'(b) = \varphi_h \text{ for all } a \in L_{h+1} \text{ and } b \in L_h, h = 1, \ldots, s-1, & (11.4)\\
\varphi_h/\varphi_{h'} \geqslant \gamma \text{ if } e_h > e_{h'}, \ h, h' = 1, \ldots, s-1, & (11.5)\\
\sum_{a \in A^R} \left(\sigma^+(a) + \sigma^-(a)\right) \leqslant \overline{\sigma} & (11.6)\\
\varphi_h \geqslant 0, h = 1, \ldots, s-1, & (11.7)\\
\sigma^+(a) \geqslant 0, \ \sigma^-(a) \geqslant 0 \text{ for all } a \in A^R. & (11.8)
\end{array}\right\} E_{Ratio}$$

The value of $\overline{\sigma}$ is obtained solving the following auxiliary optimization problem

$$\overline{\sigma} = \min \sum_{a \in A^R} (\sigma^+(a) + \sigma^-(a)), \quad \text{subject to,} \quad E'_{Ratio}, \tag{12}$$

where E'_{Ratio} is obtained from E_{Ratio} replacing constraints (11.5) and (11.6) with the following one

$$\varphi_h \geqslant \varphi_{h'} \text{ if } e_h \geqslant e_{h'}, h, h' = 1, \ldots, s-1.$$

Even if the regression problem (11) is generally not linear, it can be handled by means of the many available nonlinear programming solvers. In the following, we shall refer to (9) and (11) as difference-based and ratio-based regression models, respectively. Observe that in regression problems (9) and (11) one can consider information about intensity of preferences expressed in terms of the DCM, 1-9 AHP scale or 0-6 MACBETH scale. However, we could consider information expressed in terms of intensity of preference of a merely ordinal type such as "a is preferred to b at least as strongly as c is preferred to d", without considering the predetermined scale expressed in terms of cards, 1-9 AHP scale or 0-6 MACBETH scale. This type of preference information has been discussed in ordinal regression and ROR in Figueira et al. (2009).

Example 2.1 (continuation). *Applying the difference-based LP regression model (9) to the preference information over regions* $\texttt{R}_\texttt{i}$, $i = 1, \ldots, 4$, *collected through the DCM and considering a value function expressed in terms of a weighted sum, we get* $\delta_1 = \delta_3 = 30$ *and* $\delta_2 = \delta_4 = 13.75$. *Reminding that* $e_1 = e_3 = 5$ *and* $e_2 = e_4 = 2$, *this means that five and two blank cards have a value of 30 and 13.75, respectively. Solving the difference-based LP regression model (9) one gets a null total sum of the errors*

($\overline{\sigma} = 0$) and the following weights of criteria $w_1 = 0.5, w_2 = 0.125, w_3 = 0.375$, so that, the overall evaluations of regions are obtained as listed in Table 10.

Table 10: Scores assigned to regions by the value function $U(\cdot)$ obtained solving the difference-based LP problem (9)

Regions	$U(\cdot)$	$U'(\cdot)$	$\sigma^+(\cdot)$	$\sigma^-(\cdot)$
R_1	*87.5*	*87.5*	*0*	*0*
R_2	*73.75*	*73.75*	*0*	*0*
R_3	*43.75*	*43.75*	*0*	*0*
R_4	*30.00*	*30.00*	*0*	*0*

Table 11: Scores assigned to regions by the value function $U(\cdot)$ obtained solving the ratio-based regr. problem (11)

Regions	$U(\cdot)$	$U'(\cdot)$	$\sigma^+(\cdot)$	$\sigma^-(\cdot)$
R_1	*94.37*	*94.37*	*0*	*0*
R_2	*71.03*	*71.03*	*0*	*0*
R_3	*47.55*	*47.55*	*0*	*0*
R_4	*35.79*	*35.79*	*0*	*0*

Applying the ratio-based model (11), one gets $\varphi_1 = \varphi_3 = 1.49$ and $\delta_2 = \delta_4 = 1.33$ with $\gamma = 1.12$. Reminding that $e_1 = e_3 = 5$ and $e_2 = e_4 = 2$, this means that five and two blank cards correspond to a ratio 1.33 and 1.12, respectively. Solving the ratio-based regression model (11) one gets a null total sum of the errors ($\overline{\sigma} = 0$) and the following weights of criteria $w_1 = 0.21, w_2 = 0.61, w_3 = 0.18$, so that, the overall evaluations of regions are obtained as listed in Table 11. ■

In the following, we will refer to the standard regression problem (1) based on the DCM. Of course, any extension we propose in the next sections can be generalized to the other regression models discussed above.

3. A more robust and richer DOR

In this section, we shall extend the DOR approach described in the previous section aiming to provide more robust recommendations on the considered problem and to take into account hierarchical structures of criteria.

3.1. Robust Ordinal Regression

Solving the LP problems (1) and (3) one finds a value function $\overline{U}$ that is compatible or deviates as little as possible from the DM's preferences. However, taking into account robustness concerns, it is reasonable to consider other value functions close to $\overline{U}$ obtained by some perturbation of its parameters. In this perspective, we shall consider *compatible with the DM's preferences* any value function U satisfying the constraints $E^{DM'}$ (with $k > 0$), so that, it is reasonable taking into account all of them. With this aim we apply the ROR (Greco et al., 2008; Corrente et al., 2013), defining a necessary and a possible preference relation on the set of alternatives A. Given $a_{j_1}, a_{j_2} \in A$, on the one hand, a_{j_1} is necessarily (weakly) preferred to a_{j_2}, denoted by $a_{j_1} \succsim^N a_{j_2}$, if a_{j_1} is at least as good as a_{j_2} for all compatible value functions, while, on the other hand, a_{j_1} is possibly (weakly) preferred to a_{j_2}, denoted by $a_{j_1} \succsim^P a_{j_2}$, if a_{j_1} is at least as good as a_{j_2} for at least one compatible value function.

To compute the necessary and possible preference relations, the following LP problems need to be solved for each pair of alternatives $(a_{j_1}, a_{j_2}) \in A \times A$:

$$\varepsilon^P(a_{j_1}, a_{j_2}) = \max \varepsilon, \text{ subject to} \quad \left.\begin{array}{l} U(a_{j_1}) \geqslant U(a_{j_2}), \\ E^{DM''} \end{array}\right\} E^P(a_{j_1}, a_{j_2}) \tag{13}$$

$$\varepsilon^N(a_{j_1}, a_{j_2}) = \max \varepsilon, \text{ subject to} \quad \left.\begin{array}{l} U(a_{j_2}) \geqslant U(a_{j_1}) + \varepsilon, \\ E^{DM''}. \end{array}\right\} E^N(a_{j_1}, a_{j_2}) \tag{14}$$

where $E^{DM''}$ is obtained by $E^{DM'}$ replacing $k \geqslant 0$ with $k \geqslant \varepsilon$. In particular:

Case P1) $E^P(a_{j_1}, a_{j_2})$ is infeasible or $\varepsilon^P(a_{j_1}, a_{j_2}) \leqslant 0$: then, there is not any compatible value function for which $U(a_{j_1}) \geqslant U(a_{j_2})$. Consequently, *not*$(a_{j_1} \succsim^P a_{j_2})$ (implying that $a_{j_2} \succsim^N a_{j_1}$),

Case P2) $E^P(a_{j_1}, a_{j_2})$ is feasible and $\varepsilon^P(a_{j_1}, a_{j_2}) > 0$: then, there is at least one compatible value function such that $U(a_{j_1}) \geqslant U(a_{j_2})$ and, therefore, $a_{j_1} \succsim^P a_{j_2}$;

Case N1) $E^N(a_{j_1}, a_{j_2})$ is infeasible or $\varepsilon^N(a_{j_1}, a_{j_2}) \leqslant 0$: then, there is not any compatible value function for which $U(a_{j_2}) > U(a_{j_1})$. Therefore, $a_{j_1} \succsim^N a_{j_2}$,

Case N2) $E^N(a_{j_1}, a_{j_2})$ is feasible and $\varepsilon^N(a_{j_1}, a_{j_2}) > 0$, then, there is at least one compatible value function for which $U(a_{j_2}) > U(a_{j_1})$. Therefore, *not*$(a_{j_1} \succsim^N a_{j_2})$ (implying that $a_{j_2} \succsim^P a_{j_1}$).

3.2. Stochastic Multicriteria Acceptability Analysis

Even if in a different way in comparison with the ROR, the SMAA (Lahdelma et al., 1998; Pelissari et al., 2020) provides recommendations on the alternatives at hand considering the whole space of compatible value functions.

SMAA gives information in statistical terms based on a set of compatible value functions sampled from the simplex defined by constraints in $E^{DM''}$. Because all the above constraints are linear, one can efficiently perform the sampling using the Hit-And-Run (HAR) method (Smith, 1984; Van Valkenhoef et al., 2014). In our context, for each sampled value function, a ranking of the alternatives in A can be obtained. Based on these rankings, SMAA provides the following indices:

- *Rank Acceptability Index* (RAI), $b^v(a)$: it is the frequency with which $a \in A$ is in place v, $v = 1, \ldots, |A|$, in the considered rankings,
- *Pairwise Winning Index* (PWI), $p(a_{j_1}, a_{j_2})$: it is the frequency with which a_{j_1} is preferred to a_{j_2}.

Even if RAIs provide more robust information on the considered problem, in general, they do not give a total ranking of the alternatives under consideration. To overcome this problem, two different procedures can be used:

- *Computing the expected ranking of each alternative:* Following Lahdelma and Salminen (2001), each $a \in A$ can be associated to a value $ER(a)$ being the weighted average of its RAIs. Formally, $ER(a) = \sum_{v=1}^{|A|} v \cdot b^v(a)$. On the basis of $ER(a)$, one can obtain a complete ranking of the alternatives in A so that, for each $a, b \in A$, a is ranked not worse than b if $ER(a) \leqslant ER(b)$;
- *Computing the ranking obtained by the barycenter of the compatible space:* As mentioned earlier, the SMAA indices are calculated by considering the rankings of the alternatives, which are determined by using each value function sampled from the space defined by constraints in $E^{DM''}$. The value function obtained by averaging the sampled value functions has been proved to be able to well represent the preferences of the DM (Arcidiacono et al., 2023). Therefore, one can use this compatible value function, which represents an approximation of the barycenter of the space defined by constraints in $E^{DM''}$, to rank the alternatives at hand.

3.3. Multiple Criteria Hierarchy Process

In real-world decision making problems criteria are generally structured in a hierarchical way. It is possible to consider a root criterion (the objective of the problem), and some macro-criteria descending from it until the elementary criteria placed at the bottom of the hierarchy and on which the alternatives are evaluated. The Multiple Criteria Hierarchy Process (MCHP Corrente et al. 2012) can then be used to deal

with such problems. It permits to take into account the preferences of the DM at both partial and global levels as well as providing recommendations globally and partially.
According to the MCHP framework, in the following, $g_{\mathbf{r}}$ will be a generic criterion in the hierarchy, while by $g_{\mathbf{r}} = g_{\mathbf{0}}$ we refer to the whole set of criteria; I_G is the set of the indices of all criteria in the hierarchy; $EL \subseteq I_G$ is the set of the indices of elementary criteria, while, $E(g_{\mathbf{r}}) \subseteq EL$ is the set of indices of the elementary criteria descending from $g_{\mathbf{r}}$; the value of an alternative a on criterion $g_{\mathbf{r}}$ will be denoted by $U_{\mathbf{r}}(a)$, while the global value of a will be denoted by $U_{\mathbf{0}}(a)$. The value of a on macro-criterion $g_{\mathbf{r}}$, that is, $U_{\mathbf{r}}(a)$ will depend only on its performance on the elementary criteria descending from it. Of course, the definition of $U_{\mathbf{r}}$ will change according to the type of function used to represent the DM's preferences (see Appendix A).

Extending our proposal described in Section 2 to the MCHP framework, the DM is therefore asked (but they are not obliged) to apply the DCM for each macro-criterion $g_{\mathbf{r}}$ following these steps:

1. Rank-ordering the alternatives from the less preferred to the most preferred with respect to criterion $g_{\mathbf{r}}$ in sets $L_{(\mathbf{r},1)}, L_{(\mathbf{r},2)}, \ldots, L_{(\mathbf{r},s(\mathbf{r}))}$. That is, alternatives in $L_{(\mathbf{r},h+1)}$ are preferred to alternatives in $L_{(\mathbf{r},h)}$ on $g_{\mathbf{r}}$ for all $h = 1, \ldots, s(\mathbf{r}) - 1$ and alternatives in $L_{(\mathbf{r},h)}$ are indifferent on $g_{\mathbf{r}}$, for all $h = 1, \ldots, s(\mathbf{r})$;
2. Putting a certain number of blank cards $e_{(\mathbf{r},h)}$ between sets $L_{(\mathbf{r},h)}$ and $L_{(\mathbf{r},h+1)}$ to increase the difference between the value on $g_{\mathbf{r}}$ of the alternatives in $L_{(\mathbf{r},h+1)}$ and the value on $g_{\mathbf{r}}$ of the alternatives in $L_{(\mathbf{r},h)}$;
3. Providing the number of blank cards $e_{(\mathbf{r},0)}$ between the "fictitious zero alternative on $g_{\mathbf{r}}$" and the alternatives in $L_{(\mathbf{r},1)}$. In this case fictitious zero alternative on $g_{\mathbf{r}}$ is a fictitious alternative a_0 having a null value on $g_{\mathbf{r}}$ ($U_{\mathbf{r}}(a_0) = 0$);
4. Each alternative $a \in A^R$ is assigned a value $\nu_{\mathbf{r}}(a)$ for each macro-criterion $g_{\mathbf{r}}$ such that if $a \in L_{(\mathbf{r},h+1)}$ and $a' \in L_{(\mathbf{r},h)}$, $\nu_{\mathbf{r}}(a) = \nu_{\mathbf{r}}(a') + (e_{(\mathbf{r},h)} + 1)$, $h = 1, \ldots, s(\mathbf{r}) - 1$ and if $a \in L_{(\mathbf{r},1)}$ then $\nu_{\mathbf{r}}(a) = e_{(\mathbf{r},0)} + 1$. Consequently, for $a \in L_{(\mathbf{r},h)}$, we have: $\nu_{\mathbf{r}}(a) = \sum_{p=0}^{h-1} \left(e_{(\mathbf{r},p)} + 1\right)$;
5. The parameters of the value function U are determined in a way that for all $a \in A^R$ and all $g_{\mathbf{r}}$, $U_{\mathbf{r}}(a)$ deviates as less as possible from $k_{\mathbf{r}} \cdot \nu_{\mathbf{r}}(a)$ with $k_{\mathbf{r}}$ a scalarizing constant that can be interpreted as the value of a single blank card w.r.t. macro-criterion $g_{\mathbf{r}}$. More precisely, for all $a \in A^R$ and all macro-criteria $g_{\mathbf{r}}$, one considers the positive and negative deviations $\sigma_{\mathbf{r}}^{+}(a)$ and $\sigma_{\mathbf{r}}^{-}(a)$ between $U_{\mathbf{r}}(a)$ and $k_{\mathbf{r}} \cdot \nu_{\mathbf{r}}(a)$, that is $U_{\mathbf{r}}(a) - \sigma_{\mathbf{r}}^{+}(a) + \sigma_{\mathbf{r}}^{-}(a) = k_{\mathbf{r}} \cdot \nu_{\mathbf{r}}(a)$.

The preference information the DM provides on macro-criterion $g_{\mathbf{r}}$ through the DCM can be translated into the following set of constraints

$$\left.\begin{array}{l}
U_{\mathbf{r}}(a) \geqslant U_{\mathbf{r}}(a'), \text{ for all } a, a' \in A^R, \text{ such that } \nu_{\mathbf{r}}(a) \geqslant \nu_{\mathbf{r}}(a'), \\
U_{\mathbf{r}}(a) - \sigma_{\mathbf{r}}^{+}(a) + \sigma_{\mathbf{r}}^{-}(a) = k_{\mathbf{r}} \cdot \nu_{\mathbf{r}}(a) \;\text{ for all } a \in A^R, \\
k_{\mathbf{r}} \geqslant 0, \\
\sigma_{\mathbf{r}}^{+}(a) \geqslant 0, \; \sigma_{\mathbf{r}}^{-}(a) \geqslant 0 \text{ for all } a \in A^R,
\end{array}\right\} E_{\mathbf{r}}$$

where:

- $U_{\mathbf{r}}(a)$ is the value assigned by U to alternative a w.r.t. macro-criterion $g_{\mathbf{r}}$. The formalization of $U_{\mathbf{r}}$ depends on the type of value function U used to approximate the preferences of the DM (see Section Appendix A),

- $k_{\mathbf{r}}$ is the value of a blank card on $g_{\mathbf{r}}$,

- $\sigma^{+}_{\mathbf{r}}(a)$ and $\sigma^{-}_{\mathbf{r}}(a)$ are over and under estimations related to a and $g_{\mathbf{r}}$.

To check a function U compatible with the DM's preferences and presenting the minimum error, one has to solve the following problem:

$$\overline{\sigma}_{MCHP}=\min\sum_{\mathbf{r}\in I_G\setminus EL}\sum_{a\in A^R}\left(\sigma^{+}_{\mathbf{r}}(a)+\sigma^{-}_{\mathbf{r}}(a)\right),\;\;\text{subject to,}$$

$$\left.\begin{array}{l}E^{Model},\\ \cup_{\mathbf{r}\in I_G\setminus EL}E_{\mathbf{r}}.\end{array}\right\}E^{DM}_{MCHP}$$

To ensure that the value of each blank card $k_{\mathbf{r}}$ is greater than zero one has to solve iteratively the following LP problem:

$$\left.\begin{array}{l}\varepsilon^{*}_{MCHP}=\max\varepsilon,\;\;\text{subject to,}\\ E^{DM'}_{MCHP},\\ \displaystyle\sum_{\mathbf{r}\in I_G\setminus EL}\sum_{a\in A^R}\left(\sigma^{+}_{\mathbf{r}}(a)+\sigma^{-}_{\mathbf{r}}(a)\right)\leqslant\overline{\sigma}_{MCHP}+\eta\left(\overline{\sigma}_{MCHP}\right),\end{array}\right\}E^{DM''}_{MCHP}\qquad(15)$$

where $E^{DM'}_{MCHP}$ is obtained by E^{DM}_{MCHP} replacing the constraints $k_{\mathbf{r}}\geqslant 0$ (one in each $E_{\mathbf{r}}$) with $k_{\mathbf{r}}\geqslant\varepsilon$ and $\eta\left(\overline{\sigma}_{MCHP}\right)$ is an admitted deterioration error with respect to the optimal value obtained in the previous step, that is, $\overline{\sigma}_{MCHP}$. At the beginning, $\eta\left(\overline{\sigma}_{MCHP}\right)=0$. However, if $\varepsilon^{*}_{MCHP}=0$, then, one has to increase $\eta\left(\overline{\sigma}_{MCHP}\right)$ (as suggested in Section 2) until $\varepsilon^{*}_{MCHP}>0$.
Let us observe that, differently from LP problem (3) where we have maximized directly the value of the unique blank card k, here, we have replaced $k_{\mathbf{r}}\geqslant 0$ with $k_{\mathbf{r}}\geqslant\varepsilon$ and, then, we have maximized ε to ensure that if $\varepsilon^{*}_{MCHP}>0$ then all $k_{\mathbf{r}}$ are greater than zero.
To conclude this section, let us observe the following:

- The DM is not obliged to provide the preference information on all macro-criteria in the hierarchy but only on those they are more confident. Moreover, some information can also be imprecisely given (see Section 4),

- ROR can be applied in the MCHP context. In particular, to check if $a_{j_1}\succsim^{P}_{\mathbf{r}}a_{j_2}$ (a_{j_1} is possibly preferred to a_{j_2} on $g_{\mathbf{r}}$) or $a_{j_1}\succsim^{N}_{\mathbf{r}}a_{j_2}$ (a_{j_1} is necessarily preferred to a_{j_2} on $g_{\mathbf{r}}$), one has to solve the following two LP problems:

$$\left.\begin{array}{l}\varepsilon^{P}_{\mathbf{r}}(a_{j_1},a_{j_2})=\max\varepsilon,\;\;\text{subject to}\\ U_{\mathbf{r}}(a_{j_1})\geqslant U_{\mathbf{r}}(a_{j_2}),\\ E^{DM''}_{MCHP}.\end{array}\right\}E^{P}_{\mathbf{r}}(a_{j_1},a_{j_2})\qquad(16)$$

$$\left.\begin{array}{l}\varepsilon^{N}_{\mathbf{r}}(a_{j_1},a_{j_2})=\max\varepsilon,\;\;\text{subject to}\\ U_{\mathbf{r}}(a_{j_2})\geqslant U_{\mathbf{r}}(a_{j_1})+\varepsilon,\\ E^{DM''}_{MCHP}.\end{array}\right\}E^{N}_{\mathbf{r}}(a_{j_1},a_{j_2})\qquad(17)$$

  As commented in Section 3.1,

  - $a_{j_1}\succsim^{P}_{\mathbf{r}}a_{j_2}$ iff $E^{P}_{\mathbf{r}}(a_{j_1},a_{j_2})$ is feasible and $\varepsilon^{P}_{\mathbf{r}}(a_{j_1},a_{j_2})>0$;
  - $a_{j_1}\succsim^{N}_{\mathbf{r}}a_{j_2}$ iff $E^{N}_{\mathbf{r}}(a_{j_1},a_{j_2})$ is infeasible or $\varepsilon^{N}_{\mathbf{r}}(a_{j_1},a_{j_2})\leqslant 0$;

- SMAA can be applied in the MCHP context on the basis of a sampling of value functions from the simplex defined by constraints in $E^{DM''}_{MCHP}$. For each sampled value function, a ranking of the alternatives at hand can be done on each macro-criterion $g_{\mathbf{r}}$, $\mathbf{r} \in I_G \setminus EL$. Consequently, the RAIs and PWIs defined in Section 3.2 can be computed for each macro-criterion $g_{\mathbf{r}}$:

- $b^v_{\mathbf{r}}(a)$: it is the frequency with which alternative $a \in A$ is in place v, $v = 1, \ldots, |A|$, in the rankings produced on $g_{\mathbf{r}}$,
- $p_{\mathbf{r}}(a_{j_1}, a_{j_2})$: it is the frequency with which a_{j_1} is preferred to a_{j_2} on $g_{\mathbf{r}}$.

Analogously, the expected ranking of each alternative a on $g_{\mathbf{r}}$ $(ER_{\mathbf{r}}(a_j))$ as well as the ranking obtained using the approximation of the barycenter of the space defined by constraints in $E^{DM''}_{MCHP}$ can be computed. Formally, for each $g_{\mathbf{r}}$, $\mathbf{r} \in I_G \setminus EL$, $ER_{\mathbf{r}}(a) = \displaystyle\sum_{v=1}^{|A|} v \cdot b^v_{\mathbf{r}}(a)$.

4. Providing imprecise information

In this section, we shall extend the proposal described in Section 2. We shall consider the case in which the DM is not able to precisely provide the number of blank cards between two successive subsets of alternatives L_h and L_{h+1}, with $h = 1, \ldots, s-1$.

4.1. Interval information

Let us assume that the DM can define the minimum (e^L_h) and the maximum (e^U_h) number of blank cards to be included between sets L_h and L_{h+1}. Formally, this means that the number of blank cards e_h between the two subsets of alternatives is such that $e_h \in [e^L_h, e^U_h]$ for all $h = 0, \ldots, s-1$. To infer a value function U able to represent the DM's preference information expressed by the use of the "imprecise" DCM, one has to solve the following LP problem:

$$\overline{\sigma}_{Interval} = \min \sum_{a \in A^R} (\sigma^+(a) + \sigma^-(a)), \;\; \text{subject to}$$

$$\left.\begin{array}{l}
E^{Model} \\
U(a) \geqslant U(a'), \text{ for all } a \in L_h, a' \in L_{h'}, \text{ such that } h \geqslant h', h, h' = 1, \ldots, s \\
U(a) - \sigma^+(a) + \sigma^-(a) = \widehat{\nu}(a) \text{ for all } a \in A^R \\
k \geqslant 0 \\
\sigma^+(a) \geqslant 0, \; \sigma^-(a) \geqslant 0 \text{ for all } a \in A^R \\
\left.\begin{array}{l}
\widehat{\nu}(a) \geqslant \widehat{\nu}(a') + (e^L_h + 1) \cdot k, \\
\widehat{\nu}(a) \leqslant \widehat{\nu}(a') + (e^U_h + 1) \cdot k,
\end{array}\right\} \text{for all } a \in L_{h+1}, a' \in L_h, \; h = 0, \ldots, s-1,
\end{array}\right\} E^{DM}_{Interval}$$

where:

- E^{Model} is the set of monotonicity and normalization constraints related to the considered preference function U (see Appendix A);
- $U(a) \geqslant U(a')$ imposes that, following the ranking given by the DM, if $a \in L_h$ and $a' \in L_{h'}$ such that $h \geqslant h'$, the value of a is not lower than the value of a'. Observe that this constraint implies that $U(a) = U(a')$ in case both of them belong to the same level L_h, with $h = 1, \ldots, s$;

- $U(a)-\sigma^+(a)+\sigma^-(a)=\widehat{\nu}(a)$ imposes that the value of a (that is $U(a)$) is equal to $\widehat{\nu}(a)$ being implicitly given by the product between the value of a blank card (k) and the number of levels up to alternative a ($\nu(a)$), that is $\widehat{\nu}(a)=k\cdot\nu(a)$;
- $k\geqslant 0$ is the value of a single blank card;
- $\sigma^+(a)\geqslant 0, \sigma^-(a)\geqslant 0$ impose that the slack variables, representing overestimation and underestimation of $U(a)$, respectively, are not negative;
- constraints $\widehat{\nu}(a)\geqslant\widehat{\nu}(a')+(e_h^L+1)\cdot k$ and $\widehat{\nu}(a)\leqslant\widehat{\nu}(a')+(e_h^U+1)\cdot k$ link the values assigned by the DCM to alternatives a and a' taking into account the interval $[e_h^L,\ e_h^U]$ of the possible values of the blank card e_h put between two contiguous levels L_h and L_{h+1}, $h=0,\ldots,s-1$. Indeed,

$$e_h^L\leqslant e_h\leqslant e_h^U\Rightarrow\left(e_h^L+1\right)\leqslant(e_h+1)\leqslant\left(e_h^U+1\right)\Rightarrow\left(e_h^L+1\right)\cdot k\leqslant(e_h+1)\cdot k\leqslant\left(e_h^U+1\right)\cdot k\Rightarrow$$

$$\Rightarrow\widehat{\nu}\left(a'\right)+\left(e_h^L+1\right)\cdot k\leqslant\underbrace{\widehat{\nu}\left(a'\right)+(e_h+1)\cdot k}_{\widehat{\nu}(a)}\leqslant\widehat{\nu}\left(a'\right)+\left(e_h^U+1\right)\cdot k.$$

Following the description given in Section 2, the value of the blank card k has to be greater than zero. To maximize it, one has to solve the following LP problem

$$\left.\begin{array}{l}\varepsilon^*_{Interval}=\max\varepsilon,\ \text{subject to},\\ E^{DM'}_{Interval},\\ \displaystyle\sum_{a\in A^R}\left(\sigma^+(a)+\sigma^-(a)\right)\leqslant\overline{\sigma}_{Interval}+\eta\left(\overline{\sigma}_{Interval}\right)\end{array}\right\}\qquad(18)$$

where $E^{DM'}_{Interval}$ is obtained by $E^{DM}_{Interval}$ replacing the constraint $k\geqslant 0$ with $k\geqslant\varepsilon$ and $\eta\left(\overline{\sigma}_{Interval}\right)$ is an admitted deterioration error with respect to the optimal value obtained in the previous step, that is, $\overline{\sigma}_{Interval}$. At the beginning, $\eta\left(\overline{\sigma}_{Interval}\right)=0$. However, if $\varepsilon^*_{Interval}=0$, then, one has to increase $\eta\left(\overline{\sigma}_{Interval}\right)$ (as suggested in Section 2) until $\varepsilon^*_{Interval}>0$. Notice that from a computational point of view, comparing the LP problems (18) and (3), maximizing ε in (18) is equivalent to maximize k in (3).

4.2. Imprecise or missing information

In some situations, the DM could not be able to specify the lower or the upper bounds of e_h, that is, e_h^L or e_h^U. Summarizing, considering the number of blank cards e_h between levels L_h and L_{h+1}, with $h=1,\ldots,s-1$, five different cases can be observed and each of them is translated into a few linear equalities or inequalities:

A) The DM can precisely assign a number to e_h, that is, $e_h\in\mathbb{N}^*$. In this case,

$$\widehat{\nu}\left(a\right)=\widehat{\nu}(a')+(e_h+1)\cdot k,\ \text{for all } a\in L_{h+1}, a'\in L_h.$$

B) The DM can define the lower e_h^L and the upper e_h^U bound of e_h, that is, $e_h\in\left[e_h^L, e_h^U\right]$. In this case:

$$\left.\begin{array}{l}\widehat{\nu}(a)\geqslant\widehat{\nu}(a')+\left(e_h^L+1\right)\cdot k,\\ \widehat{\nu}(a)\leqslant\widehat{\nu}(a')+\left(e_h^U+1\right)\cdot k\end{array}\right\}\ \text{for all } a\in L_{h+1}, a'\in L_h.$$

C) The DM can define only the lower bound e_h^L of e_h, that is, $e_h \in \left[e_h^L, ?\right]$. In this case:

$$\widehat{\nu}(a) \geqslant \widehat{\nu}(a') + \left(e_h^L + 1\right) \cdot k, \text{ for all } a \in L_{h+1}, a' \in L_h.$$

D) The DM can define only the upper bound e_h^U of e_h, that is, $e_h \in \left[?, e_h^U\right]$. In this case:

$$\left.\begin{array}{l} \widehat{\nu}(a) \geqslant \widehat{\nu}(a') + k, \\ \widehat{\nu}(a) \leqslant \widehat{\nu}(a') + \left(e_h^U + 1\right) \cdot k \end{array}\right\} \text{ for all } a \in L_{h+1}, a' \in L_h.$$

E) The DM can define neither the lower bound e_h^L nor the upper bound e_h^U of e_h, that is, $e_h \in [?, ?]$. In this case:

$$\widehat{\nu}(a) \geqslant \widehat{\nu}(a') + k, \text{ for all } a \in L_{h+1}, a' \in L_h.$$

Taking into account all these different cases, to check for a value function compatible with the possible imprecise preference information provided by the DM, one has to solve the following LP problem:

$$\overline{\sigma}_{Imprecise} = \min \sum_{a \in A^R} (\sigma^+(a) + \sigma^-(a)), \text{ subject to}$$

$$\left.\begin{array}{l} E^{Model} \\ U(a) \geqslant U(a'), \text{ with } a \in L_h, a' \in L_{h'}, \text{ such that } h \geqslant h', \ h, h' = 1, \ldots, s-1, \\ U(a) - \sigma^+(a) + \sigma^-(a) = \widehat{\nu}(a) \ \text{ for all } a \in A^R, \\ k \geqslant 0, \\ \sigma^+(a) \geqslant 0, \ \sigma^-(a) \geqslant 0 \text{ for all } a \in A^R, \\ \left.\begin{array}{ll} \widehat{\nu}(a) = \widehat{\nu}(a') + (e_h + 1) \cdot k & \text{if } \ e_h \in \mathbb{N}^*, \\ \left.\begin{array}{l} \widehat{\nu}(a) \geqslant \widehat{\nu}(a') + \left(e_h^L + 1\right) \cdot k \\ \widehat{\nu}(a) \leqslant \widehat{\nu}(a') + \left(e_h^U + 1\right) \cdot k \end{array}\right\} & \text{if } \ e_h \in \left[e_h^L, e_h^U\right] \\ \widehat{\nu}(a) \geqslant \widehat{\nu}(a') + \left(e_h^L + 1\right) \cdot k & \text{if } \ e_h \in \left[e_h^L, ?\right] \\ \left.\begin{array}{l} \widehat{\nu}(a) \geqslant \widehat{\nu}(a') + k \\ \widehat{\nu}(a) \leqslant \widehat{\nu}(a') + \left(e_h^U + 1\right) \cdot k \end{array}\right\} & \text{if } \ e_h \in \left[?, e_h^U\right] \\ \widehat{\nu}(a) \geqslant \widehat{\nu}(a') + k & \text{if } \ e_h \in [?, ?] \end{array}\right\} \text{for all } a \in L_{h+1}, a' \in L_h. \end{array}\right\} E^{DM}_{Imprecise}$$

To ensure that the value of the blank card k is greater than zero, one has to solve the following LP problem

$$\begin{array}{l} \varepsilon^*_{Imprecise} = \max \varepsilon, \text{ subject to,} \\ \left.\begin{array}{l} E^{DM'}_{Imprecise}, \\ \displaystyle\sum_{a \in A^R} \left(\sigma^+(a) + \sigma^-(a)\right) \leqslant \overline{\sigma}_{Imprecise} + \eta\left(\overline{\sigma}_{Imprecise}\right) \end{array}\right\} E^{DM''}_{Imprecise} \end{array}$$

where $E^{DM'}_{Imprecise}$ is obtained by $E^{DM}_{Imprecise}$ replacing the constraint $k \geqslant 0$ with $k \geqslant \varepsilon$ and $\eta\left(\overline{\sigma}_{Imprecise}\right)$ is an admitted deterioration error with respect to the optimal value obtained in the previous step, that is, $\overline{\sigma}_{Imprecise}$. At the beginning, $\eta\left(\overline{\sigma}_{Imprecise}\right)=0$. However, if $\varepsilon^{*}_{Imprecise}=0$, then, one has to increase $\eta\left(\overline{\sigma}_{Imprecise}\right)$ (as suggested in Section 2) until $\varepsilon^{*}_{Imprecise}>0$.

4.3. ROR, SMAA and hierarchy of criteria in presence of imprecise information

The imprecision information described in the previous section can also be taken into account in the ROR and SMAA as well as in case the problem presents criteria structured in a hierarchical way and dealt with by the MCHP:

- Regarding the ROR and, for each $(a_{j1},a_{j2})\in A\times A$, to check if $a_{j1}\succsim^{P}a_{j2}$ and if $a_{j1}\succsim^{N}a_{j2}$ one has to solve the LP problems (13) and (14), respectively, replacing $E^{DM''}$ with $E^{DM''}_{Imprecise}$,
- Regarding the SMAA, the sampling of compatible value functions has to be done from the simplex defined by constraints in $E^{DM''}_{Imprecise}$,
- Regarding the MCHP, reminding that for each macro-criterion $g_{\mathbf{r}}$, $\mathbf{r}\in I_G\setminus EL$, $e_{(\mathbf{r},h)}$ is the number of blank cards between the sets of reference alternatives $L_{(\mathbf{r},h)}$ and $L_{(\mathbf{r},h+1)}$ and that the number of subsets in which reference alternatives are ordered w.r.t. this criterion is denoted by $s(\mathbf{r})$ (see Section 3.3), we assume that the DM can provide information on the lower $e^{L}_{(\mathbf{r},h)}$ or upper $e^{U}_{(\mathbf{r},h)}$ bounds of $e_{(\mathbf{r},h)}$. Of course, all the imprecise information presented in Section 4.2 can also be taken into account in the MCHP with the corresponding constraints translating them. For each macro-criterion $g_{\mathbf{r}}$, one needs to define the following set of constraints

$$\left.\begin{array}{l}
U_{\mathbf{r}}(a)\geqslant U_{\mathbf{r}}(a'),\ \text{for all } a\in L_h, a'\in L_{h'} \text{ such that } h\geqslant h',\ h,h'=1,\ldots,s,\\
U_{\mathbf{r}}(a)-\sigma^{+}_{\mathbf{r}}(a)+\sigma^{-}_{\mathbf{r}}(a)=\widehat{\nu}_{\mathbf{r}}(a)\ \text{ for all } a\in A^R,\\
k_{\mathbf{r}}\geqslant 0,\\
\sigma^{+}_{\mathbf{r}}(a)\geqslant 0,\ \sigma^{-}_{\mathbf{r}}(a)\geqslant 0 \text{ for all } a\in A^R,\\
\left.\begin{array}{ll}
\widehat{\nu}_{\mathbf{r}}(a)=\widehat{\nu}_{\mathbf{r}}(a')+\left(e_{(\mathbf{r},h)}+1\right)\cdot k_{\mathbf{r}}, & \text{if } \ e_{(\mathbf{r},h)}\in\mathbb{N}^{*},\\
\left.\begin{array}{l}\widehat{\nu}_{\mathbf{r}}(a)\geqslant\widehat{\nu}_{\mathbf{r}}(a')+\left(e^{L}_{(\mathbf{r},h)}+1\right)\cdot k_{\mathbf{r}}\\ \widehat{\nu}_{\mathbf{r}}(a)\leqslant\widehat{\nu}_{\mathbf{r}}(a')+\left(e^{U}_{(\mathbf{r},h)}+1\right)\cdot k_{\mathbf{r}}\end{array}\right\} & \text{if } \ e_{(\mathbf{r},h)}\in\left[e^{L}_{(\mathbf{r},h)},e^{U}_{(\mathbf{r},h)}\right]\\
\widehat{\nu}_{\mathbf{r}}(a)\geqslant\widehat{\nu}_{\mathbf{r}}(a')+\left(e^{L}_{(\mathbf{r},h)}+1\right)\cdot k_{\mathbf{r}} & \text{if } \ e_{(\mathbf{r},h)}\in\left[e^{L}_{(\mathbf{r},h)},?\right]\\
\left.\begin{array}{l}\widehat{\nu}_{\mathbf{r}}(a)\geqslant\widehat{\nu}_{\mathbf{r}}(a')+k_{\mathbf{r}},\\ \widehat{\nu}_{\mathbf{r}}(a)\leqslant\widehat{\nu}_{\mathbf{r}}(a')+\left(e^{U}_{(\mathbf{r},h)}+1\right)\cdot k_{\mathbf{r}}\end{array}\right\} & \text{if } \ e_{(\mathbf{r},h)}\in\left[?,e^{U}_{(\mathbf{r},h)}\right]\\
\widehat{\nu}_{\mathbf{r}}(a)\geqslant\widehat{\nu}_{\mathbf{r}}(a')+k_{\mathbf{r}} & \text{if } \ e_{(\mathbf{r},h)}\in[?,?]
\end{array}\right\}\ \text{for all } a,a'\in A^R.
\end{array}\right\} E^{Imprecise}_{\mathbf{r}}$$

The existence of a value function compatible with the imprecise information provided by the DM is checked solving the following LP problem:

$$\begin{array}{l}
\overline{\sigma}^{Imprecise}_{MCHP}=\min\displaystyle\sum_{\mathbf{r}\in I_G\setminus EL}\sum_{a\in A^R}\left(\sigma^{+}_{\mathbf{r}}(a)+\sigma^{-}_{\mathbf{r}}(a)\right)\ \text{ subject to},\\
\left.\begin{array}{l}E^{Model},\\ \cup_{\mathbf{r}\in I_G\setminus EL}E^{Imprecise}_{\mathbf{r}}.\end{array}\right\}E^{Imprecise}_{MCHP}
\end{array}\tag{19}$$

To ensure that the value of each blank card $k_{\mathbf{r}}$ is greater than zero, one has to solve the following LP problem

$$
\left.\begin{aligned}
&\varepsilon^{*,Imprecise}_{MCHP} = \max\varepsilon, \;\; \text{subject to,}\\
&E^{Imprecise'}_{MCHP},\\
&\sum_{\mathbf{r}\in I_G\setminus EL}\sum_{a\in A^R}\left(\sigma^{+}_{\mathbf{r}}(a)+\sigma^{-}_{\mathbf{r}}(a)\right)\leqslant \overline{\sigma}^{Imprecise}_{MCHP}+\eta\left(\overline{\sigma}^{Imprecise}_{MCHP}\right)
\end{aligned}\right\} E^{Imprecise''}_{MCHP} \tag{20}
$$

where $E^{Imprecise'}_{MCHP}$ is obtained by $E^{Imprecise}_{MCHP}$ replacing $k_{\mathbf{r}} \geqslant 0$ with $k_{\mathbf{r}} \geqslant \varepsilon$ in each set of constraints $E^{Imprecise}_{\mathbf{r}}$ and $\eta\left(\overline{\sigma}^{Imprecise}_{MCHP}\right)$ is an admitted deterioration error with respect to the optimal value obtained in the previous step, that is, $\overline{\sigma}^{Imprecise}_{MCHP}$. At the beginning, $\eta\left(\overline{\sigma}^{Imprecise}_{MCHP}\right) = 0$. However, if $\varepsilon^{*,Imprecise}_{MCHP} = 0$, then, one has to increase $\eta\left(\overline{\sigma}^{Imprecise}_{MCHP}\right)$ (as suggested in Section 2) until $\varepsilon^{*,Imprecise}_{MCHP} > 0$.

To apply ROR and SMAA in this context is therefore necessary to proceed in the following way. For each macro-criterion $g_{\mathbf{r}}$, $\mathbf{r} \in I_G \setminus EL$,

- for each $(a_{j_1}, a_{j_2}) \in A \times A$, to check if $a_{j_1} \succsim^{P}_{\mathbf{r}} a_{j_2}$ and if $a_{j_1} \succsim^{N}_{\mathbf{r}} a_{j_2}$ one has to solve the LP problems (16) and (17), respectively, replacing $E^{DM''}_{MCHP}$ with $E^{Imprecise''}_{MCHP}$;
- the sampling on which the SMAA indices computation is based, has to be performed from the simplex defined by constraints in $E^{Imprecise''}_{MCHP}$.

5. Didactic Example

In this section, we shall show how the proposed methodology could be applied in a real-world problem related to a large ongoing research project (https://grins.it/; https://dse.unibo.it/en/university-outreach/next-generation-e
Let us assume that the DM wants to rank Italian Regions according to three main macro-criteria, being Circular Economy ($g_{\mathbf{1}}$), Innovation-Driven Development ($g_{\mathbf{2}}$) and Smart Specialization Strategies ($g_{\mathbf{3}}$). Each of these macro-criteria is then articulated in several elementary criteria which are shown in Table 12. All elementary criteria have an increasing direction of preference (+) except "Urban Waste Generation" which has a decreasing direction of preference (−). Before applying our proposal, the data are normalized using the procedure described in Greco et al. (2019*b*). Normalized data are shown in Table 13.

In Section 5.1, we shall describe the application of the new framework in case the DM, for example a policy maker, wants to evaluate the regions on Smart Specialization only and they are able to give precise information. In Section 5.2, instead, we shall show the results of the application of the method in case the DM wants to rank the regions considering the whole set of criteria and they are not able to provide precise information both at global and partial levels.

5.1. Basic Framework

At first, let us assume the DM wants to evaluate the Regions only according to the Smart Specialization macro-criterion and that they can express their preferences about 5 regions only, namely $A^R = \{$Veneto, Friuli-Venezia Giulia, Marche, Liguria, Molise$\} = \{a^R_5, a^R_4, a^R_3, a^R_2, a^R_1\}$ articulated as shown in Table 14 . To avoid any confusion, we denoted by $a^R_1, \ldots, a^R_5$ the reference regions and we added in the first column the ID of the same alternative in Table 12. Moreover, in all tables regarding the whole set of regions, we put in bold the data corresponding to the reference regions.

Table 12: Regions' performances on elementary criteria divided in three Macro-Criteria: Circular Economy (g_1), Innovation-Driven Development (g_2), Smart Specialization (g_3)

		Circular Economy - $g_{(1)}$			**Innovation-Driven Development - $g_{(2)}$**			**Smart Specialization - $g_{(3)}$**		
ID	**Region**	Urban Waste Generation (kg/capita) (2017) ($g_{(1,1)}$)	Differentiated waste collection (%) (2017) ($g_{(1,2)}$)	Renewable electricity (percentage) (2017) ($g_{(1,3)}$)	Research and Development Personnel (full-time equivalents per thousand inhabitants) (2017) ($g_{(2,1)}$)	Patents registered at the European Patent Office (EPO) (number per million inhabitants) (2012) ($g_{(2,2)}$)	Number of companies with ISO 14001 certification (2017) ($g_{(2,3)}$)	Percentage of Smart specialized companies (2018) ($g_{(3,1)}$)	Percentage of value added by Smart specialised companies to the total value added in the region (2018) ($g_{(3,2)}$)	Number of Smart specialized companies with high intensity of investments in social and environmental responsibility (2018) ($g_{(3,3)}$)
1	Piedmont	470.69	59.25	35.5	7.04	90.97	1563	32.9	65.5	1040
2	Aosta Valley	582.58	61.14	243.5	3.11	51.09	102	26.1	49.5	31
3	**Liguria**	**531.68**	**48.81**	**7.3**	**5.02**	**56.67**	**672**	**29.2**	**49.2**	**522**
4	Lombardy	467.25	69.61	21.7	6.57	92.35	3581	32.8	62.4	2637
5	Trentino-South Tyrol	487.24	71.58	116.7	6.35	88.17	524	30.7	55	248
6	**Veneto**	**475.88**	**73.65**	**21.3**	**6.62**	**100.96**	**2105**	**35.6**	**61**	**1065**
7	**Friuli-Venezia Giulia**	**484.11**	**65.48**	**23.3**	**6.73**	**216.43**	**585**	**34.9**	**63.9**	**250**
8	Emilia-Romagna	642.54	63.83	19.2	9.49	131.52	1898	33.5	64.7	999
9	Tuscany	600	53.88	39.2	6.06	63.97	1403	30.5	60.2	1138
10	Umbria	508.39	61.69	37.2	4.4	33.04	395	33.2	51.9	230
11	**Marche**	**532.27**	**63.25**	**27**	**5.29**	**58.18**	**586**	**32.6**	**58.2**	**331**
12	Lazio	503.97	45.52	13.2	6.51	23	1430	30.7	67	1046
13	Abruzzo	452.52	55.99	44.6	3.46	19.2	488	32.9	53.1	258
14	**Molise**	**376.96**	**30.72**	**84.4**	**3.37**	**2.93**	**101**	**34.5**	**49.6**	**71**
15	Campania	439.06	52.76	26.4	3.28	9.64	1158	32.7	54.4	1244
16	Apulia	462.6	40.44	52.5	2.36	9.43	815	35.5	54.7	742
17	Basilicata	345.17	45.29	90.1	2.35	10.29	238	36.1	61.9	152
18	Calabria	394.61	39.67	72.6	1.8	9.13	325	38.2	50.4	487
19	Sicily	456.01	21.69	25.1	1.84	4.33	780	33.5	50.2	792
20	Sardinia	438.29	63.05	36	2.36	5.64	300	31.1	45.5	309

Table 13: Normalized performances. (+) and (−) denote elementary criteria having a positive or negative direction of preference, respectively.

		Circular Economy - g_1			**Innovation-Driven Development - g_2**			**Smart Specialization - g_3**		
ID	**Region**	Urban Waste Generation (kg/capita) (2017) (-) ($g_{(1,1)}$)	Differentiated waste collection (%) (2017) (+) ($g_{(1,2)}$)	Renewable electricity (percentage) (2017) (+) ($g_{(1,3)}$)	Research and Development Personnel (full-time equivalents per thousand inhabitants) (2017) (+) ($g_{(2,1)}$)	Patents registered at the European Patent Office (EPO) (number per million inhabitants) (2012) (+) ($g_{(2,2)}$)	Number of companies with ISO 14001 certification (2017) (+) ($g_{(2,3)}$)	Percentage of Smart specialized companies (2018) (+) ($g_{(3,1)}$)	Percentage of value added by Smart specialised companies to the total value added in the region (2018) (+) ($g_{(3,2)}$)	Number of Smart specialized companies with high intensity of investments in social and environmental responsibility (2018)(+) ($g_{(3,3)}$)
1	Piedmont	0.53	0.56	0.45	0.68	0.62	0.62	0.5	0.74	0.6
2	Aosta Valley	0.26	0.58	1	0.37	0.49	0.33	0.07	0.32	0.32
3	**Liguria**	**0.38**	**0.43**	**0.36**	**0.52**	**0.51**	**0.44**	**0.27**	**0.31**	**0.46**
4	Lombardy	0.54	0.69	0.4	0.65	0.62	1	0.5	0.66	1
5	Trentino-South Tyrol	0.49	0.71	0.71	0.63	0.61	0.41	0.36	0.46	0.38
6	**Veneto**	**0.52**	**0.74**	**0.4**	**0.65**	**0.65**	**0.73**	**0.67**	**0.62**	**0.61**
7	**Friuli-Venezia Giulia**	**0.5**	**0.64**	**0.41**	**0.66**	**1**	**0.43**	**0.63**	**0.7**	**0.38**
8	Emilia-Romagna	0.12	0.62	0.4	0.88	0.74	0.69	0.54	0.72	0.59
9	Tuscany	0.22	0.49	0.46	0.61	0.53	0.59	0.35	0.6	0.63
10	Umbria	0.44	0.59	0.45	0.48	0.43	0.39	0.52	0.38	0.37
11	**Marche**	**0.38**	**0.61**	**0.42**	**0.55**	**0.51**	**0.43**	**0.48**	**0.55**	**0.4**
12	Lazio	0.45	0.39	0.38	0.64	0.4	0.6	0.36	0.78	0.6
13	Abruzzo	0.57	0.52	0.48	0.4	0.39	0.41	0.5	0.41	0.38
14	**Molise**	**0.75**	**0.21**	**0.6**	**0.4**	**0.34**	**0.33**	**0.6**	**0.32**	**0.33**
15	Campania	0.6	0.48	0.42	0.39	0.36	0.54	0.49	0.45	0.66
16	Apulia	0.55	0.33	0.5	0.31	0.36	0.47	0.67	0.45	0.52
17	Basilicata	0.82	0.39	0.62	0.31	0.36	0.36	0.71	0.64	0.35
18	Calabria	0.71	0.32	0.57	0.27	0.36	0.37	0.84	0.34	0.45
19	Sicily	0.56	0.1	0.41	0.27	0.34	0.47	0.54	0.34	0.53
20	Sardinia	0.6	0.61	0.45	0.32	0.35	0.37	0.39	0.21	0.4

Table 14: DM's preferences

ID	Region	Level
6	Veneto	$\nu(a_5^R) = 65$
		$e_{(\mathbf{3},4)} = 8$
7	Friuli-Venezia Giulia	$\nu(a_4^R) = 56$
		$e_{(\mathbf{3},3)} = 6$
11	Marche	$\nu(a_3^R) = 49$
		$e_{(\mathbf{3},2)} = 5$
3	Liguria	$\nu(a_2^R) = 43$
		$e_{(\mathbf{3},1)} = 5$
14	Molise	$\nu(a_1^R) = 37$
		$e_{(\mathbf{3},0)} = 36$

Table 15: Basic Framework - Parameters' values obtained solving the LP problem (1) assuming the weighted sum as preference model

$w_{(\mathbf{3},\mathbf{1})}$	$w_{(\mathbf{3},\mathbf{2})}$	$w_{(\mathbf{3},\mathbf{3})}$	$k_{\mathbf{3}}$	$\sigma^-_{(\mathbf{3},5)}$	$\sigma^+_{(\mathbf{3},5)}$	$\sigma^-_{(\mathbf{3},4)}$	$\sigma^+_{(\mathbf{3},4)}$	$\sigma^-_{(\mathbf{3},3)}$	$\sigma^+_{(\mathbf{3},3)}$	$\sigma^-_{(\mathbf{3},2)}$	$\sigma^+_{(\mathbf{3},2)}$	$\sigma^-_{(\mathbf{3},1)}$	$\sigma^+_{(\mathbf{3},1)}$	$\overline{\sigma}$
0.099	0.41	0.492	0.01	0	0	0	0	0.002	0	0	0.032	0	0	0.035

Following the description in Section 2, according to the provided information, *Molise* is the worst among the five reference regions, while *Liguria* is the best among them. Moreover, 36 ($e_{(\mathbf{3},0))}$) blank cards are included between the "fictitious zero alternative" and *Molise*, 5 ($e_{(\mathbf{3},1)}$) blank cards between *Molise* and *Liguria* and so on. Solving the LP problem (1) and considering the Weighted Sum as preference model we get the parameters shown in Table 15.

As one can see, the LP problem is not able to capture the DM's preferences without any estimation errors since $\overline{\sigma} = 0.035$. For such a reason, we try to represent the same preferences using the 2-additive Choquet integral. Solving the same LP problem, we get that the Choquet integral is able to represent the DM's preferences without any errors since $\overline{\sigma} = 0$. At this point, we solve the LP problem (3) to maximize the value of the blank card obtaining the Möbius parameters shown in Table 16.

As described in Section 3.1, to get more robust recommendations on the considered problem, we can compute the possible and necessary preference relations which are shown in Tables 17 and 18, respectively. Referring to Table 17 entry 1 means that the region in the row is possibly preferred to the region in the column, while entry 0 means that the region in the row is not possibly preferred to the region in the column (analogous interpretations can be given to the entries in Table 18). Looking at the two preference relations, it is possible to state that:

Table 16: Basic Framework - Parameters' values obtained solving the LP problem (3) assuming the 2-additive Choquet integral as preference model

$m(\{g_1\})$	$m(\{g_2\})$	$m(\{g_3\})$	$m(\{g_1,g_2\})$	$m(\{g_1,g_3\})$	$m(\{g_2,g_3\})$	k	$\overline{\sigma}$
0.0953	0.3561	0.7061	0.0712	-0.0953	-0.1334	0.0095	0

Table 17: Basic Framework - Possible preference relation between regions on Smart Specialization

ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	**0**	**1**	**1**	**0**	**1**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**1**
4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	0	1	0	0	1	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1
6	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
7	**0**	**1**	**1**	**0**	**1**	**0**	**1**	**0**	**0**	**1**	**1**	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**
8	0	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
9	0	1	1	0	1	0	1	0	1	1	1	0	1	1	1	1	1	1	1	1
10	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1
11	**0**	**1**	**1**	**0**	**1**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	**1**	**1**	**0**	**0**	**0**	**1**	**0**	**1**
12	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
13	0	1	0	0	1	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1
14	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**1**
15	0	1	1	0	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1
16	0	1	1	0	1	0	0	0	0	1	1	0	1	1	0	1	1	1	1	1
17	0	1	1	0	1	0	0	0	0	1	1	0	1	1	0	0	1	1	1	1
18	0	1	1	0	1	0	0	0	0	1	1	0	1	1	0	0	0	1	0	1
19	0	1	1	0	1	0	0	0	0	1	1	0	1	1	0	0	1	1	1	1
20	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1

- Lombardy ($ID = 4$) is necessarily preferred to all regions and, therefore, it can be considered the best among them;
- Piedmont ($ID = 1$), can be considered the third region since it is necessarily preferred to all regions apart from Lombardy and Lazio ($ID = 12$); Emilia-Romagna ($ID = 8$) is necessarily preferred to all regions apart from Piedmont, Lombardy and Lazio, while Lazio is necessarily preferred to all regions apart from Piedmont, Lombardy, Emilia-Romagna, Veneto ($ID = 8$) and Campania ($ID = 15$);
- Aosta Valley ($ID = 2$), is the worst region since it is not possibly preferred to any region (apart from itself);
- Molise ($ID = 14$) is possibly preferred to Aosta Valley and Sardinia ($ID = 20$), only, and, analogously, Sardinia is possibly preferred to Aosta Valley and Molise only.

Since there is often no clear preference between many pairs of alternatives, we chose to use the SMAA as described in Section 3.2. The SMAA application is based on a sampling of 1 million compatible measures computing, therefore, the RAIs and the PWIs shown in Tables 19 and 20, respectively. Analyzing these indices, it is possible to notice that:

- Aosta Valley ($ID = 2$) is always ranked last ($b_{\mathbf{3}}^{20}(a_2) = 100\%$) while Lombardy ($ID = 4$) is always ranked first ($b_{\mathbf{3}}^{1}(a_4) = 100\%$);
- Second to last and third to last positions are taken by Molise ($ID = 14$) and Sardinia ($ID = 20$). In particular, Molise is most frequently in the second to last position ($b_{\mathbf{3}}^{19}(a_{14}) = 72.13\%$), while, Sardinia is in the third to last position with the same frequency ($b_{\mathbf{3}}^{18}(a_{20}) = 72.13\%$);
- Piedmont ($ID = 1$) is always in the second and third positions ($b_{\mathbf{3}}^{2}(a_1) = 79.75\%$ and $b_{\mathbf{3}}^{3}(a_1) = 20.25\%$), Emilia-Romagna ($ID = 8$) fills the third or the fourth position in the whole ranking ($b_{\mathbf{3}}^{3}(a_8) = 50.96\%$

Table 18: Basic Framework - Necessary preference relation between regions on Smart Specialization

ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	**0**	**1**	**1**	**0**	**1**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**1**
4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	0	1	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1
6	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
7	**0**	**1**	**1**	**0**	**1**	**0**	**1**	**0**	**0**	**1**	**1**	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**
8	0	1	1	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
9	0	1	1	0	1	0	1	0	1	1	1	0	1	1	0	1	1	1	1	1
10	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1
11	**0**	**1**	**1**	**0**	**1**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**1**
12	0	1	1	0	1	0	1	0	1	1	1	1	1	1	0	1	1	1	1	1
13	0	1	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1
14	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**
15	0	1	1	0	1	0	1	0	0	1	1	0	1	1	1	1	1	1	1	1
16	0	1	1	0	1	0	0	0	0	1	1	0	1	1	0	1	1	1	1	1
17	0	1	1	0	1	0	0	0	0	1	1	0	1	1	0	0	1	1	0	1
18	0	1	1	0	1	0	0	0	0	1	0	0	1	1	0	0	0	1	0	1
19	0	1	1	0	1	0	0	0	0	1	1	0	1	1	0	0	0	1	1	1
20	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 19: Basic Framework - Rank Acceptability Indices (RAIs) on Smart Specialization

Rank / ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	79.75	20.25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100
3	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**
4	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	44.94	55.06	0	0	0	0
6	**0**	**0**	**0**	**32.44**	**67.56**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
7	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
8	0	0	50.96	49.04	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	10.05	89.95	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0	0	0
11	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**73.73**	**26.27**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
12	0	20.25	28.79	18.51	19.04	13.4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	55.06	44.94	0	0	0	0
14	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**27.87**	**72.13**	**0**
15	0	0	0	0	13.4	76.55	10.05	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	84.3	15.7	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	26.27	73.73	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	15.7	84.3	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	72.13	27.87	0

Table 20: Basic Framework - Pairwise Winning Indices (PWIs) on Smart Specialization

ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	100	100	0	100	100	100	100	100	100	100	79.75	100	100	100	100	100	100	100	100
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	**0**	**100**	**0**	**0**	**100**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**100**
4	100	100	100	0	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
5	0	100	0	0	0	0	0	0	0	100	0	0	44.94	100	0	0	0	0	0	100
6	**0**	**100**	**100**	**0**	**100**	**0**	**100**	**0**	**100**	**100**	**100**	**32.44**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
7	**0**	**100**	**100**	**0**	**100**	**0**	**0**	**0**	**0**	**100**	**100**	**0**	**100**	**100**	**0**	**100**	**100**	**100**	**100**	**100**
8	0	100	100	0	100	100	100	0	100	100	100	50.96	100	100	100	100	100	100	100	100
9	0	100	100	0	100	0	100	0	0	100	100	0	100	100	10.05	100	100	100	100	100
10	0	100	0	0	0	0	0	0	0	0	0	0	0	100	0	0	0	0	0	100
11	**0**	**100**	**100**	**0**	**100**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**100**	**100**	**0**	**0**	**0**	**73.73**	**0**	**100**
12	20.25	100	100	0	100	67.56	100	49.04	100	100	100	0	100	100	86.6	100	100	100	100	100
13	0	100	0	0	55.06	0	0	0	0	100	0	0	0	100	0	0	0	0	0	100
14	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**27.87**
15	0	100	100	0	100	0	100	0	89.95	100	100	13.4	100	100	0	100	100	100	100	100
16	0	100	100	0	100	0	0	0	0	100	100	0	100	100	0	0	100	100	100	100
17	0	100	100	0	100	0	0	0	0	100	100	0	100	100	0	0	0	100	84.3	100
18	0	100	100	0	100	0	0	0	0	100	26.27	0	100	100	0	0	0	0	0	100
19	0	100	100	0	100	0	0	0	0	100	100	0	100	100	0	0	15.7	100	0	100
20	0	100	0	0	0	0	0	0	0	0	0	0	0	72.13	0	0	0	0	0	0

and $b^4_{\mathbf{3}}(a_8) = 49.04\%$), while Veneto ($ID = 6$) is always in fourth or fifth position ($b^4_{\mathbf{3}}(a_6) = 32.44\%$ and $b^5_{\mathbf{3}}(a_6) = 67.56\%$);

- Tuscany ($ID = 7$), Trentino-South Tyrol ($ID = 16$), Emilia Romagna ($ID = 3$) and Basilicata ($ID = 10$) are always set on ranks 8, 9, 14 and 17, respectively.

Considering the PWIs, the data in Table 20 show that in most of the cases, there is an evident preference for one alternative over the other since $p_{\mathbf{r}}(a, b) = 100\%$ and, therefore, $p_{\mathbf{r}}(b, a) = 0\%$. The only doubtful cases are the following:

- Trentino-South Tyrol ($ID = 5$) is preferred to Abruzzo ($ID = 13$) with a frequency of 44.94%, while the vice versa is true for 55.06% of the cases;

- Emilia-Romagna ($ID = 8$) and Lazio ($ID = 12$) are preferred one to the other with similar frequencies since $p_{\mathbf{3}}(8, 12) = 50.96\%$, while $p_{\mathbf{3}}(12, 8) = 49.04\%$.

In all other cases, one region is preferred to the other with a frequency not lower than 67.56%.

Let us observe that the barycenter shown in Table 22 is computed as the average, component by component, of the sampled measures. The first observation is that the two rankings are the same. On the one hand, considering the top of the ranking, Lombardy is in the first position, followed by Piedmont, while Emilia-Romagna and Lazio, which had similar frequencies to be preferred one to the other, are third and fourth, respectively. On the other hand, looking at the bottom of the same ranking, the last three places are taken by Sardinia, Molise, and Aosta Valley.

Commenting on the barycenter shown in Table 22, taking into account the weights assigned to single criteria without considering interactions, one can observe that Number of Smart specialized companies with high intensity of investments in social and environmental responsibility (2018) ($g_{(\mathbf{3},\mathbf{3})}$) has a greater weight than

Table 21: Basic Framework - Expected Ranking and ranking obtained using the approximation of the barycenter shown in Table 22

(a) Expected Ranking

Rank	Region	ID	$ER(\cdot)$
1	Lombardy	4	100
2	Piedmont	1	220.25
3	Emilia-Romagna	8	349.04
4	Lazio	12	376.55
5	**Veneto**	**6**	**467.56**
6	Campania	15	596.64
7	Tuscany	9	689.95
8	**Friuli-Venezia Giulia**	**7**	**800**
9	Apulia	16	900
10	Basilicata	17	1015.7
11	Sicily	19	1084.3
12	**Marche**	**11**	**1226.27**
13	Calabria	18	1273.73
14	**Liguria**	**3**	**1400**
15	Abruzzo	13	1544.94
16	Trentino-South Tyrol	5	1555.06
17	Umbria	10	1700
18	Sardinia	20	1827.87
19	**Molise**	**14**	**1872.13**
20	Aosta Valley	2	2000

(b) Ranking based on barycenter parameters

Rank	Region	ID	$U(a_j)$
1	Lombardy	4	0.88213
2	Piedmont	1	0.63429
3	Emilia-Romagna	8	0.62726
4	Lazio	12	0.62696
5	**Veneto**	**6**	**0.61956**
6	Campania	15	0.60514
7	Tuscany	9	0.58144
8	**Friuli-Venezia Giulia**	**7**	**0.53377**
9	Apulia	16	0.51954
10	Basilicata	17	0.51014
11	Sicily	19	0.49586
12	**Marche**	**11**	**0.46705**
13	Calabria	18	0.46239
14	**Liguria**	**3**	**0.40986**
15	Abruzzo	13	0.40613
16	Trentino-South Tyrol	5	0.40601
17	Umbria	10	0.39053
18	Sardinia	20	0.36098
19	**Molise**	**14**	**0.35267**
20	Aosta Valley	2	0.27889

Table 22: Basic Framework - Parameters of the approximated barycenter

$m(\{g_{(\mathbf{3},\mathbf{1})}\})$	$m(\{g_{(\mathbf{3},\mathbf{2})}\})$	$m(\{g_{(\mathbf{3},\mathbf{3})}\})$	$m(\{g_{(\mathbf{3},\mathbf{1})}, g_{(\mathbf{3},\mathbf{2})}\})$	$m(\{g_{(\mathbf{3},\mathbf{1})}, g_{(\mathbf{3},\mathbf{3})}\})$	$m(\{g_{(\mathbf{3},\mathbf{2})}, g_{(\mathbf{3},\mathbf{3})}\})$
0.0918	0.3540	0.7306	0.0768	-0.0100	-0.2432

Percentage of value added by Smart specialized companies to the total value added in the region (2018) ($g_{(\mathbf{3},\mathbf{2})}$) that, in turn, has a greater weight than Percentage of Smart specialized companies (2018) ($g_{(\mathbf{3},\mathbf{1})}$). Moreover, there is a positive interaction between $g_{(\mathbf{3},\mathbf{1})}$ and $g_{(\mathbf{3},\mathbf{2})}$ $\left(m\left(\{g_{(\mathbf{3},\mathbf{1})}, g_{(\mathbf{3},\mathbf{2})}\}\right) = 0.0768\right)$, while there is a negative interaction between $g_{(\mathbf{3},\mathbf{1})}$ and $g_{(\mathbf{3},\mathbf{3})}$ $\left(m\left(\{g_{(\mathbf{3},\mathbf{1})}, g_{(\mathbf{3},\mathbf{3})}\}\right) = -0.0100\right)$ as well as between $g_{(\mathbf{3},\mathbf{2})}$ and $g_{(\mathbf{3},\mathbf{3})}$ $\left(m\left(\{g_{(\mathbf{3},\mathbf{2})}, g_{(\mathbf{3},\mathbf{3})}\}\right) = -0.2432\right)$.

5.2. Imprecise Information

Let's suppose in this case that the DM provides their preferences on the same 5 regions above considered both globally (g_0) and on the single macro-criteria (Circular Economy - g_1, Innovation-Driven Development - g_2, Smart Specialization - g_3). Moreover, for each macro-criterion, the DM provides interval or imprecise information (see Section 4.2). Let us assume that the preference is articulated as shown in Table 23 and let us comment on it.

Considering the information in Table 23a, the DM retains that, at the global level, Molise is the least preferred region, followed by Liguria, Marche, Friuli-Venezia Giulia and, finally, Veneto, being the most preferred. On the one hand, the DM provides imprecise information regarding the number of blank cards between the zero fictitious level and Molise $\left(\left[e^L_{(\mathbf{0},\mathbf{0})},\ e^R_{(\mathbf{0},\mathbf{0})}\right] = [40, 50]\right)$, the number of blank cards between Liguria and Marche $\left(\left[e^L_{(\mathbf{0},\mathbf{2})},\ e^R_{(\mathbf{0},\mathbf{2})}\right] = [1, 6]\right)$ and the number of blank cards between Friuli-Venezia Giulia and Veneto $\left(\left[e^L_{(\mathbf{0},\mathbf{4})}, e^U_{(\mathbf{0},\mathbf{4})}\right] = [1, 7]\right)$. On the other hand, the DM gives missing information on the number of blank cards between Molise and Liguria since $e^L_{(\mathbf{0},\mathbf{1})}$ =? and for the number of blank cards between Marche

Table 23: Imprecise Information - DM's preference at global and macro-criteria level

(a) Global level - $g_{(\mathbf{0})}$

ID	Region	Level
6	Veneto	$e_{(\mathbf{0},4)} \in \left[e^{L}_{(\mathbf{0},4)}, e^{U}_{(\mathbf{0},4)}\right] = [1, 7]$
7	Friuli-Venezia Giulia	$e_{(\mathbf{0},3)} \in \left[e^{L}_{(\mathbf{0},3)},\ e^{U}_{(\mathbf{0},3)}\right] = [7, ?]$
11	Marche	$e_{(\mathbf{0},2)} \in \left[e^{L}_{(\mathbf{0},2)},\ e^{U}_{(\mathbf{0},2)}\right] = [1, 6]$
3	Liguria	$e_{(\mathbf{0},1)} \in \left[e^{L}_{(\mathbf{0},1)},\ e^{U}_{(\mathbf{0},1)}\right] = [?, 5]$
14	Molise	$e_{(\mathbf{0},0)} \in \left[e^{L}_{(\mathbf{0},0)},\ e^{U}_{(\mathbf{0},0)}\right] = [40, 50]$

(b) Circular Economy - $g_{(\mathbf{1})}$

ID	Region	Level
6	Veneto	$e_{(\mathbf{1},4)} \in \left[e^{L}_{(\mathbf{1},4)}, e^{U}_{(\mathbf{1},4)}\right] = [1, 6]$
7	Friuli-Venezia Giulia	$e_{(\mathbf{1},3)} \in \left[e^{L}_{(\mathbf{1},3)}, e^{U}_{(\mathbf{1},3)}\right] = [?, 4]$
11	Marche	$e_{(\mathbf{1},2)} \in \left[e^{L}_{(\mathbf{1},2)}, e^{U}_{(\mathbf{1},2)}\right] = [1, 3]$
14	Molise	$e_{(\mathbf{1},1)} \in \left[e^{L}_{(\mathbf{1},1)}, e^{U}_{(\mathbf{1},1)}\right] = [?, 5]$
3	Liguria	$e_{(\mathbf{1},0)} \in \left[{}^{L}_{(\mathbf{1},0)}, e^{U}_{(\mathbf{1},0)}\right] = [9, 16]$

(c) Innovation-Driven - $g_{(\mathbf{2})}$

ID	Region	Level
7	Friuli-Venezia Giulia	$e_{(\mathbf{2},4)} \in \left[e^{L}_{(\mathbf{2},4)}, e^{U}_{(\mathbf{2},4)}\right] = [1, 5]$
6	Veneto	$e_{(\mathbf{2},3)} \in \left[e^{L}_{(\mathbf{2},3)}, e^{U}_{(\mathbf{2},3)}\right] = [2, 7]$
11	Marche	$e_{(\mathbf{2},2)} \in \left[e^{L}_{(\mathbf{2},2)}, e^{U}_{(\mathbf{2},2)}\right] = [?, 5]$
3	Liguria	$e_{(\mathbf{2},1)} \in \left[e^{L}_{(\mathbf{2},1)}, e^{U}_{(\mathbf{2},1)}\right] = [1, 6]$
14	Molise	$e_{(\mathbf{2},0)} \in \left[e^{L}_{(\mathbf{2},0)}, e^{U}_{(\mathbf{2},0)}\right] = [10, 20]$

(d) Smart Specialization - $g_{(\mathbf{3})}$

ID	Region	Level
6	Veneto	$e_{(\mathbf{3},4)} \in \left[e^{L}_{(\mathbf{3},4)}, e^{U}_{(\mathbf{3},4)}\right] = [1, 6]$
7	Friuli-Venezia Giulia	$e_{(\mathbf{3},3)} \in \left[e^{L}_{(\mathbf{3},3)}, e^{U}_{(\mathbf{3},3)}\right] = [3, ?]$
11	Marche	$e_{(\mathbf{3},2)} \in \left[e^{L}_{(\mathbf{3},2)}, e^{U}_{(\mathbf{3},2)}\right] = [2, 5]$
3	Liguria	$e_{(\mathbf{3},1)} \in \left[e^{L}_{(\mathbf{3},1)}, e^{U}_{(\mathbf{3},1)}\right] = [?, 7]$
14	Molise	$e_{(\mathbf{3},0)} \in \left[e^{L}_{(\mathbf{3},0)}, e^{U}_{(\mathbf{3},0)}\right] = [10, 18]$

and Friuli-Venezia Giulia since $e^{U}_{(\mathbf{0},\mathbf{3})}$ =?. The information contained in Tables 23b-23d can be interpreted analogously. One thing that should be underlined is that using the MCHP within the proposed framework (see Section 4.3) the DM can provide different information on the considered macro-criteria. For example, while the five reference regions are ordered in the same way at the global level (see Table 23a) and on Smart

Specialization (see Table 23d), the same does not happen for the other two macro-criteria. Indeed, while Liguria is preferred to Molise at the global level, the vice versa is true on Circular Economy (see Table 23b). Analogously, while Veneto is preferred to Friuli-Venezia Giulia at the global level, the vice versa is true on Innovation-Driven Development (see Table 23c). This permits the DM to give more detailed information not only at the global but also at the partial one.

Solving the LP problem (19) and assuming the weighted sum as preference model, we find $\overline{\sigma} = 0$. Therefore, this time, the weighted sum can represent the preferences given by the DM. Solving the LP problem (20) maximizing the blank cards' values, we get the weights of the elementary criteria shown in Table 24 as well as the number of blank cards and the value of one blank card (globally and on all macro-criteria) shown in Table 25.

Table 24: Imprecise Information - Elementary criteria weights obtained solving LP problem (20) and assuming the weighted sum as preference model

$w_{(\mathbf{1},1)}$	$w_{(\mathbf{1},2)}$	$w_{(\mathbf{1},3)}$	$w_{(\mathbf{2},1)}$	$w_{(\mathbf{2},2)}$	$w_{(\mathbf{2},3)}$	$w_{(\mathbf{3},1)}$	$w_{(\mathbf{3},2)}$	$w_{(\mathbf{3},3)}$
0.097	0.119	0	0.385	0.047	0.01	0.033	0.159	0.149

As one can see, the number of blank cards to be included between two successive levels is perfectly compatible with the DM's preferences. For example, while the DM stated that $e_{(\mathbf{0},4)} \in [1, 7]$ (see Table 23a), the number of blank cards obtained solving the LP problem is $e_{(\mathbf{0},4)} = 1.1604$ (see Table 25). Analogously, while the DM stated that $e_{(\mathbf{3},3)} \in [3, ?]$ (see Table 23d), the value obtained solving the previously mentioned LP problem is $e_{(\mathbf{3},3)} = 3$ (see Table 25). Considering, instead, the weights obtained for the considered elementary criteria, it seems that Renewable Electricity (percentage) (2017) has not any importance since $w_{(\mathbf{1},3)} = 0$, while, the most important elementary criterion is Research and Development Personnel (full-time equivalents per thousand inhabitants) (2017) ($w_{(\mathbf{2},1)} = 0.385$), followed by Percentage of value added by Smart Specialized companies to the total value added in the region (2018) ($w_{(\mathbf{3},2)} = 0.159$) and Number of Smart Specialized companies with high intensity of investments in social and environmental responsibility (2018) ($w_{(\mathbf{3},3)} = 0.149$).

To get more robust recommendations on the problem at hand, we computed the ROR and SMAA methodologies. To save space, we do not report here all the corresponding tables that are provided, instead, as supplementary material. Let us comment on the results obtained by SMAA.

Considering the RAIs at the global and partial levels, the following can be observed:

Table 25: Imprecise Information - Number of blank cards and value of a single blank card both at the global and partial levels

$g_{(\mathbf{0})}$		$g_{(\mathbf{1})}$		$g_{(\mathbf{2})}$		$g_{(\mathbf{3})}$	
$e_{(\mathbf{0},4)}$	1.1604	$e_{(\mathbf{1},4)}$	1.2139	$e_{(\mathbf{2},4)}$	1	$e_{(\mathbf{3},4)}$	2.7161
$e_{(\mathbf{0},3)}$	10.2139	$e_{(\mathbf{1},3)}$	1.2674	$e_{(\mathbf{2},3)}$	4.9015	$e_{(\mathbf{3},3)}$	3
$e_{(\mathbf{0},2)}$	6	$e_{(\mathbf{1},2)}$	1	$e_{(\mathbf{2},2)}$	0	$e_{(\mathbf{3},2)}$	4.8209
$e_{(\mathbf{0},1)}$	5	$e_{(\mathbf{1},1)}$	0.3637	$e_{(\mathbf{2},1)}$	6	$e_{(\mathbf{3},1)}$	0
$e_{(\mathbf{0},0)}$	40	$e_{(\mathbf{1},0)}$	13.0545	$e_{(\mathbf{2},0)}$	19.3095	$e_{(\mathbf{3},0)}$	18
$k_{\mathbf{0}}$	0.0095	$k_{\mathbf{1}}$	0.0063	$k_{\mathbf{2}}$	0.0085	$k_{\mathbf{3}}$	0.0063

Table 26: Imprecise Information - Comparison between RAIs computed at global and partial levels by the MCHP: Focus on Lombardy ($ID = 4$) and Trentino-South Tyrol ($ID = 5$)

Rank / ID		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
4	$g_{(\mathbf{0})}$	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$g_{(\mathbf{1})}$	0	61.07	38.93	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$g_{(\mathbf{2})}$	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$g_{(\mathbf{3})}$	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	$g_{(\mathbf{0})}$	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0
	$g_{(\mathbf{1})}$	0	38.93	61.07	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$g_{(\mathbf{2})}$	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$g_{(\mathbf{3})}$	0	0	0	0	0	0	0	0	0	0	0	0	22.39	77.61	0	0	0	0	0	0

- At the global level, Lombardy ($ID = 4$) is always at the first position ($b^1_{\mathbf{0}}(a_4) = 100\%$), while, Sicily ($ID = 19$) is always in the last rank position ($b^{20}_{\mathbf{0}}(a_{19}) = 100\%$);
- On Circular Economy, Veneto ($ID = 6$) is always first ($b^1_{\mathbf{1}}(a_6) = 100\%$), while Sicily is always at the bottom of the ranking ($b^{20}_{\mathbf{1}}(a_{19}) = 100\%$);
- On Innovation-Driven Development, Emilia-Romagna ($ID = 8$) is robustly in the first position ($b^1_{\mathbf{2}}(a_8) = 100\%$), while Calabria ($ID = 18$) is always in the last rank position ($b^{20}_{\mathbf{2}}(a_{18}) = 100\%$);
- On Smart Specialization, Lombardy is in the first rank position in all considered cases ($b^1_{\mathbf{3}}(a_4) = 100\%$), while the last position is taken by Aosta Valley ($ID = 2$) and Sardinia ($ID = 20$) with frequencies 90.19% and 9.81%, respectively.

Even if the previous analysis permits to appreciate the recommendations on the considered regions at the partial level, let us underline even more this aspect considering the data in Table 26. In the table, we report the RAIs at the global and partial levels for two regions, that is, Lombardy ($ID = 4$) and Trentino-South Tyrol ($ID = 5$). As one can see, while at the global level and on Smart Specialization, Lombardy is always in the first rank position, the same cannot be said for Circular Economy and Innovation-Driven Development. On the one hand, considering Circular Economy, Lombardy is in the second or third positions with frequencies of 61.07% and 38.93%, respectively, while, on the other hand, it is always fifth on Innovation-Driven Development.

The different recommendations provided by the introduced framework are even more evident considering Trentino-South Tyrol. Indeed, while at the global level, it is always in the seventh position, on Circular Economy it is between the second and the third positions ($b^2_{\mathbf{1}}(a_5) = 38.93\%$) and ($b^3_{\mathbf{1}}(a_5) = 61.07\%$), it is always in the sixth position on Innovation-Driven Development, and, finally, it is between positions 13th and 14th on Smart Specialization with frequencies of 22.39% and 77.61%, respectively.

To summarize the results of the RAIs, let us show in Table 27 the expected rankings both at the global and partial levels.

Looking at the data in the tables, one has the confirmation of the goodness of Lombardy being in the first rank-position at the global level and on Smart Specialization, while, it is second-ranked on Circular Economy and fifth-ranked on Innovation-Driven Development. Analogously, looking at the bottom of the rankings, Sicily performs badly at the global level and on Circular Economy being placed in the last position

Table 27: Imprecise Information - Expected rankings of the considered regions both at the global and partial levels

(a) Global level - $g_{(\mathbf{0})}$

Rank	Region	ID	$ER(\cdot)$
1	Lombardy	4	100
2	Emilia-Romagna	8	200
3	Piedmont	1	345.76
4	**Veneto**	**6**	**354.24**
5	**Friuli-Venezia Giulia**	**7**	**500**
6	Lazio	12	600
7	Trentino-South Tyrol	5	700
8	Tuscany	9	800
9	**Marche**	**11**	**900**
10	Campania	15	1000
11	Umbria	10	1100
12	Basilicata	17	1226.97
13	**Liguria**	**3**	**1273.03**
14	Abruzzo	13	1400
15	Apulia	16	1500
16	**Molise**	**14**	**1608.87**
17	Sardinia	20	1691.13
18	Calabria	18	1834.82
19	Aosta Valley	2	1865.18
20	Sicily	19	2000

(b) Circular economy - $g_{(\mathbf{1})}$

Rank	Region	ID	$ER(\cdot)$
1	**Veneto**	**6**	**100**
2	Lombardy	4	238.93
3	Trentino-South Tyrol	5	261.07
4	Sardinia	20	400
5	Basilicata	17	500
6	**Friuli-Venezia Giulia**	**7**	**600**
7	Piedmont	1	700
8	Abruzzo	13	800
9	Campania	15	900
10	Umbria	10	1000
11	**Marche**	**11**	**1100**
12	Calabria	18	1200
13	Aosta Valley	2	1348.36
14	**Molise**	**14**	**1351.64**
15	Apulia	16	1500
16	Lazio	12	1600
17	**Liguria**	**3**	**1700**
18	Emilia-Romagna	8	1800
19	Tuscany	9	1900
20	Sicily	19	2000

(c) Innovation-Driven - $g_{(\mathbf{2})}$

Rank	Region	ID	$ER(\cdot)$
1	Emilia-Romagna	8	100
2	**Friuli-Venezia Giulia**	**7**	**200**
3	Piedmont	1	300
4	**Veneto**	**6**	**400**
5	Lombardy	4	500
6	Trentino-South Tyrol	5	600
7	Lazio	12	700
8	Tuscany	9	800
9	**Marche**	**11**	**900**
10	**Liguria**	**3**	**1000**
11	Umbria	10	1100
12	Abruzzo	13	1200
13	**Molise**	**14**	**1310.59**
14	Campania	15	1389.44
15	Aosta Valley	2	1499.97
16	Apulia	16	1600
17	Basilicata	17	1700
18	Sardinia	20	1800
19	Sicily	19	1900
20	Calabria	18	2000

(d) Smart Specialization - $g_{(\mathbf{3})}$

Rank	Region	ID	$ER(\cdot)$
1	Lombardy	4	100
2	Lazio	12	200
3	Piedmont	1	300
4	Emilia-Romagna	8	400
5	**Veneto**	**6**	**500**
6	Tuscany	9	600
7	**Friuli-Venezia Giulia**	**7**	**700**
8	Campania	15	800
9	Basilicata	17	900
10	Apulia	16	1000
11	**Marche**	**11**	**1100**
12	Sicily	19	1200
13	Calabria	18	1322.39
14	Trentino-South Tyrol	5	1377.61
15	Abruzzo	13	1500
16	Umbria	10	1600
17	**Liguria**	**3**	**1700**
18	**Molise**	**14**	**1800**
19	Sardinia	20	1909.81
20	Aosta Valley	2	1990.19

but also on Innovation-Driven Development since it is in the second to last position. However, it is not so badly ranked on Smart Specialization since it is placed in the 12th rank position.

Regarding some other regions, a floating behavior can be observed. For example, Trentino-South Tyrol is seventh at the global level, while, it is third on Circular Economy, sixth on Innovation-Driven Development and fourteenth on Smart Specialization. This means that, while Circular Economy can be considered a strong point of the region, Smart Specialization can be considered a weak point deserving improvement. Analogously, Sardinia, is seventeenth at the global level, while it is ranked fourth on Circular economy, eighteenth on Innovation-Driven Development and nineteenth on Smart Specialization.
This information can be used by policymakers and political governments to evaluate which are the weak and strong points of each region and, consequently, build politics aiming to improve the weak points by pushing on the strong points.

6. Conclusions

Preference parameter elicitation procedures are fundamental to Multiple Criteria Decision Aiding (MCDA) and must take into account two main requirements: on the one hand, the preference information requested from the Decision Maker (DM) must be as simple and easy as possible, and on the other hand, it must be as rich and precise as possible. In this perspective, we considered the newly introduced Deck of cards based Ordinal Regression (DOR) (Barbati et al., 2024) that permits to collect preference information not only in terms of ranking of reference alternatives but also of intensity of preferences. This can be done through the Deck of Cards Method (DCM) that, in general, is perceived as a relatively intuitive and straightforward method and, as discussed at length in this paper, also with other well-known MCDA procedures such as AHP, BWM and MACBETH. In this context, we took into consideration some issues that are particularly relevant in real-world applications: robustness concerns; imprecision, incompleteness and ill determination of the preference information; and hierarchy of considered criteria. With this aim, Robust Ordinal Regression, Stochastic Multicriteria Acceptability Analysis, and Multiple Criteria Hierarchy Process were extended to DOR. Observe that the proposed approach puts together two main research streams in MCDA, that is, on the one hand, the scaling procedures related to DCM, AHP and MACBETH, and, on the other hand, the ordinal regression with its extensions such as Robust Ordinal Regression. The resulting methodology is very flexible and versatile. Indeed, according to the requirements and the previous experiences of the DM, one can select 1) the most appropriate scaling procedure such as DCM, AHP, BWM, MACBETH or other analogous methodologies acquiring and processing DM's pairwise preferences comparisons between reference alternatives; 2) the most adequate formal model of a value function, e.g. weighted sum, piecewise additive value function, Choquet integral, assigning an overall evaluation to each alternative; 3) the most satisfactory form to represent preferences on the set of alternatives taking into account imprecision and indetermination in the preference parameters, e.g. possible and necessary preferences, pairwise preferences probability or probability of attaining a given ranking. This so rich and ductile methodology can be successfully applied to complex real-world decision problems characterized by (i) many heterogeneous criteria structured in a hierarchical way; (ii) vague and approximate preference information; (iii) plurality of experts, stakeholders, policymakers and, in general, diversified actors participating to the decision process. We have shown the potential of the proposed methodology in a didactic example related to the ranking of Italian Regions with respect to Circular Economy, Innovation-Driven Development and Smart Specialization Strategies. The example is based on a large ongoing national project financed by the European Union - NextGenerationEU where Italian Regions are evaluated on several elementary criteria so that those considered in our example are just a small subset. Let us underline that the methodology can be applied across various contexts such as environmental or human development and potentially in the construction of composite indicators in any domain (for an updated state-of-the-art survey on composite indicators see Greco et al. 2019*a*).

For the future, we plan to apply the proposed methodology to some real-world decision problems aiming, on the one hand, to test its effectiveness, reliability and validity and, on the other hand, to develop some customized procedures and protocols for applications in specific domains. Related to this, we would like to test the advantages of the proposed methodology in some decision experiments that could unveil spaces for further improvements. A further domain for future research is the extension of DOR to sorting decision problems in which the alternatives, rather than ordered from best to worst, have to be assigned to predefined ordered classes. We would like also to explore a possible extension of DOR to outranking methods such as ELECTRE (Figueira et al., 2013; Govindan and Jepsen, 2016) and PROMETHEE (Behzadian et al., 2010; Brans and Vincke, 1985).

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Appendix A. Different preference models

The procedure described in Sections 2 - 4, proposing to apply the DCM to build a value function compatible with a few DM's preferences, is independent of the type of the assumed preference model U. For this reason, in the following, we shall recall four different types of value functions that can be used under the proposed framework to build a compatible value function, namely, a weighted sum, a piecewise linear value function (Jacquet-Lagreze and Siskos, 1982), a general additive value function (Greco et al., 2008) and the Choquet integral value function (Choquet, 1953; Grabisch, 1996). The choice of the considered value function will determine the constraints composing the E^{Model} set mentioned at first in the LP problem (1) included in Section 2 and, therefore, present in the subsequent mathematical problems. The four preference functions are described in the following.

Weighted sum: denoting by $w_1, \ldots, w_n$ the weights of criteria $g_1, \ldots, g_n$, E^{Model} is the following set of constraints:

$$\left.\begin{array}{l} U(a) = \displaystyle\sum_{i=1}^{n} w_i \cdot g_i(a) \text{ for all } a \in A, \\ w_i \geqslant 0, \text{ for all } i = 1, \ldots, n, \\ \displaystyle\sum_{i=1}^{n} w_i = 1. \end{array}\right\} E^{Model}_{WS}$$

Under the MCHP framework, the value assigned by U to alternative a on macro-criterion $g_{\mathbf{r}}$ is $U_{\mathbf{r}}(a) = \sum\limits_{\mathbf{t} \in E(g_{\mathbf{r}})} w_{\mathbf{t}} \cdot g_{\mathbf{t}}(a)$ where $w_{\mathbf{t}}$ is the weight attached to the elementary criterion $g_{\mathbf{t}}$ and they are such that $w_{\mathbf{t}} \geqslant 0$ for all $\mathbf{t} \in EL$. In this case, the parameters defining the value function U are the weights $w_{\mathbf{t}}$, $\mathbf{t} \in EL$;

Piecewise linear value function: in this case, the value function U is such that $U(a) = \sum\limits_{i=1}^{n} u_i(g_i(a))$ for all $a \in A$ and each $u_i : A \rightarrow \mathbb{R}$ is a piecewise linear value function for each $g_i \in G$. Assuming that all criteria are expressed on a quantitative scale and $X_i = \{x_i^1, \ldots, x_i^{m_i}\} \subseteq \mathbb{R}$ is the set of possible values that can be obtained on criterion $g_i \in G$, each function u_i is defined by γ_i breakpoints $y_i^1, \ldots, y_i^{\gamma_i} \in \mathbb{R}$ such that $y_i^1 < \cdots < y_i^{\gamma_i}$ and

$$\left[x_i^1, x_i^{m_i}\right] = \left[y_i^1, y_i^2\right] \cup \cdots \cup \left[y_i^{\gamma_i - 1}, y_i^{\gamma_i}\right].$$

Therefore, the marginal value function u_i will be defined by means of $u_i(y_i^1), \ldots, u_i(y_i^{\gamma_i})$ only so that, if $g_i(a) \in [y_i^{q-1}, y_i^q]$, $q = 2, \ldots, \gamma_i$, we have

$$u_i(g_i(a)) = \frac{y_i^q - g_i(a)}{y_i^q - y_i^{q-1}} \cdot u_i\left(y_i^{q-1}\right) + \frac{g_i(a) - y_i^{q-1}}{y_i^q - y_i^{q-1}} \cdot u_i\left(y_i^q\right)$$

or, equivalently,

$$u_i(g_i(a)) = \frac{g_i(a) - y_i^{q-1}}{y_i^q - y_i^{q-1}} \cdot \left(u_i\left(y_i^q\right) - u_i\left(y_i^{q-1}\right)\right) + u_i(y_i^{q-1}).$$

Consequently, E^{Model} is replaced by the following set of constraints:

$$\left.\begin{array}{l} U(a)=\sum\limits_{i=1}^{n}u_i(g_i(a)) \text{ for all } a\in A, \\ u_i(y_i^{q-1})\leqslant u_i(y_i^{q}), \text{ for all } i=1,\ldots,n, \text{ and for all } q=2,\ldots,\gamma_i, \\ u_i(x_i^1)=0, \text{ for all } i=1,\ldots,n, \\ \sum\limits_{i=1}^{n}u_i\left(x_i^{m_i}\right)=1. \end{array}\right\} E_{PS}^{Model}$$

Under the MCHP framework, the value of a w.r.t. macro-criterion $g_{\mathbf{r}}$ is equal to $U_{\mathbf{r}}\left(a\right)=\sum\limits_{\mathbf{t}\in E(g_{\mathbf{r}})}u_{\mathbf{t}}\left(g_{\mathbf{t}}(a)\right)$. In this case, the set of criteria G has to be replaced by the set of elementary criteria $\{g_{\mathbf{t}},\ \mathbf{t}\in EL\}$;

General additive value function: Denoting by $X_i=\{x_i^1,\ldots,x_i^{m_i}\}\subseteq\mathbb{R}$ the set of possible values that can be obtained on criterion $g_i\in G$, the marginal value function u_i depends on $u_i(x_i^1),\ldots,u_i(x_i^{m_i})$ only. E^{Model} is replaced by the following set of constraints:

$$\left.\begin{array}{l} U(a)=\sum\limits_{i=1}^{n}u_i(g_i(a)) \text{ for all } a\in A, \\ u_i(x_i^{f-1})\leqslant u_i(x_i^{f}), \text{ for all } i=1,\ldots,n \text{ and for all } f=2,\ldots,m_i, \\ u_i(x_i^1)=0, \text{ for all } i=1,\ldots,n, \\ \sum\limits_{i=1}^{n}u_i\left(x_i^{m_i}\right)=1. \end{array}\right\} E_{GA}^{Model}$$

Under the MCHP framework the definition of $U_{\mathbf{r}}$ is the same considered when the preference model is a piecewise linear value function. The only difference w.r.t. the previous case is that for each marginal value function $u_{\mathbf{t}}$, $\mathbf{t}\in EL$, the breakpoints coincide with the evaluations of the alternatives on that elementary criterion, that is, $y_{\mathbf{t}}^1=x_{\mathbf{t}}^1$, $y_{\mathbf{t}}^2=x_{\mathbf{t}}^2$, ..., $y_{\mathbf{t}}^{\gamma_{\mathbf{t}}}=x_{\mathbf{t}}^{m_{\mathbf{t}}}$;

Choquet integral value function: as known in literature, a value function U can be written in an additive way ($U(a)=\sum\limits_{g_i\in G}u_i(g_i(a))$) iff the criteria from G are mutually preferentially independent (Wakker, 1989). However, in real-world applications, the criteria present generally a certain degree of interaction. In particular, given $g_i, g_{i'}\in G$, we say that g_i and $g_{i'}$ are positively (negatively) interacting if the importance given to them together is greater (lower) than the sum of their importance when considered alone. To take into account these interactions, non additive integrals are used in literature (Grabisch and Labreuche, 2016) and the most known is the Choquet integral. It is based on a capacity being a set function $\mu: 2^G\rightarrow[0,1]$ such that the following constraints are satisfied:

1a) $\mu(R)\leqslant\mu(S)$ for all $R\subseteq S\subseteq G$ (monotonicity),

2a) $\mu(\emptyset)=0$ and $\mu(G)=1$ (normalization).

Known μ, the Choquet integral of $(g_1(a),\ldots,g_n(a))$ is computed as follows:

$$Ch_\mu(a)=\sum_{i=1}^{n}[g_{(i)}(a)-g_{(i-1)}(a)]\cdot\mu(A_{(i)})$$

where $_{(\cdot)}$ is a permutation of the indices of criteria in G such that $0 = g_{(0)}(a) \leqslant g_{(1)}(a) \leqslant \cdots \leqslant g_{(n)}(a)$ and $A_{(i)} = \{g_{i'} \in G: \; g_{i'}(a) \geqslant g_{(i)}(a)\}$.

Since the use of the Choquet integral implies the definition of $2^{|G|} - 2$ values (one for each subset of G different from $\emptyset$ and G), in real-world applications the Möbius transformation of μ (Rota, 1964) and k-additive capacities (Grabisch, 1997) are used:

- the Möbius transformation of a capacity μ is a set function $m : 2^G \rightarrow \mathbb{R}$ such that $\mu(T) = \sum\limits_{S \subseteq T} m(S)$,
- a capacity μ is said k-additive if its Möbius transformation is such that $m(T) = 0$ for all $T \subseteq G, |T| \geqslant k$.

Because, as stated in (Grabisch, 1997) and demonstrated by several real-world applications (see, e.g., (Grabisch et al., 2002; Berrah and Clivillé, 2007; Angilella et al., 2018)), as well as in combination with other methodologies such as multi-objective optimization (see, e.g., (Branke et al., 2016)), 2-additive capacities represent a useful compromise between a fully additive but simplistic model (a weighted sum, implying independence between criteria) and a fully general but difficult-to-handle Choquet integral model (which poses challenging elicitation issues). Therefore, in the following, we consider the Choquet integral in terms of a 2-additive capacity formulated as follows:

$$U(a) = \sum_{g_i \in G} m(\{g_i\}) \cdot g_i(a) + \sum_{\{g_i, g_{i'}\} \subseteq G} m(\{g_i, g_{i'}\}) \cdot min\{g_i(a), g_{i'}(a)\}.$$

E^{Model} is therefore replaced by the following set of constraints:

$$\left.\begin{array}{l}
U(a) = \sum\limits_{g_i \in G} m(\{g_i\}) \cdot g_i(a) + \sum\limits_{\{g_i, g_{i'}\} \subseteq G} m(\{g_i, g_{i'}\}) \cdot min\{g_i(a), g_{i'}(a)\} \text{ for all } a \in A, \\
m(\{\emptyset\}) = 0, \\
\sum\limits_{g_i \in G} m(\{g_i\}) + \sum\limits_{\{g_i, g_{i'}\} \subseteq G} m(\{g_i, g_{i'}\}) = 1, \\
m(\{g_i\}) + \sum\limits_{g_i \in T} m(\{g_i, g_{i'}\}) \geqslant 0, \text{ for all } g_i \in G, \text{ and } T \subseteq G
\end{array}\right\} E^{Model}_{Choquet}$$

where the last three constraints are the equivalent of normalization (**2a)**) and monotonicity (**1a)**) constraints when a 2-additive capacity is used. Under the MCHP framework, the value of an alternative $a \in A$ w.r.t. macro-criterion $g_{\mathbf{r}}$ is given by

$$U_{\mathbf{r}}(a) = \sum_{\mathbf{t} \in E(g_{\mathbf{r}})} m\left(\{g_{\mathbf{t}}\}\right) \cdot g_{\mathbf{t}}\left(a\right) + \sum_{\{\mathbf{t}_1, \mathbf{t}_2\} \subseteq E(g_{\mathbf{r}})} m\left(\{g_{\mathbf{t}_1}, g_{\mathbf{t}_2}\}\right) \cdot \min\{g_{\mathbf{t}_1}\left(a\right), g_{\mathbf{t}_2}\left(a\right)\}.$$

Let us conclude this section observing that all mathematical programming problems discussed in the previous sections are linear because the considered preference models are affine in their parameters, that is, $U_{\alpha\pi + (1-\alpha)\pi'}(a) = \alpha \cdot U_{\pi}(a) + (1-\alpha) \cdot U_{\pi'}(a)$ for all $\alpha \in [0,1]$. In particular,

- if U is a weighted sum, then $\pi = [w_1, \ldots, w_n]$,
- if U is a piecewise linear value function, then $\pi = [u_i\left(y_i^q\right)]_{\substack{i=1,\ldots,n \\ q=2,\ldots,\gamma_i}}$,
- if U is a general additive value function, then $\pi = \left[u_i\left(x_i^f\right)\right]_{\substack{i=1,\ldots,n \\ f=2,\ldots,m_i}}$,

- If U is a 2-additive Choquet integral, then, $\pi = \left[[m(\{g_i\})]_{g_i \in G}, [m(\{g_i, g_{i'}\})]_{\{g_i, g_{i'}\} \subseteq G} \right]$.

Appendix B. Extended formulation of the LP problems to be solved in the example 2.2

- In example 2.2:

$$
\begin{aligned}
&\overline{\sigma} = \min \{\sigma^+(a) + \sigma^-(a) + \sigma^+(b) + \sigma^-(b)\}, \text{ subject to,} \\
&\left.\begin{array}{l}
\left.\begin{array}{l}
U(a) = u_1(0.3) + u_2(0.7) \\
U(b) = u_1(0.4) + u_2(0.6) \\
U(c) = u_1(0.8) + u_2(1) \\
u_1(0.3) \leqslant u_1(0.4) \leqslant u_1(0.8) \\
u_2(0.6) \leqslant u_2(0.7) \leqslant u_2(1) \\
u_1(0.3) = 0 \\
u_2(0.6) = 0 \\
u_1(0.8) + u_2(1) = 1
\end{array}\right\} E_{GA}^{Model} \\
U(a) \geqslant U(b) \\
U(a) - \sigma^+(a) + \sigma^-(a) = k \cdot 100 \\
U(b) - \sigma^+(b) + \sigma^-(b) = k \cdot 70 \\
k \geqslant 0 \\
\sigma^+(a),\ \sigma^-(a),\ \sigma^+(b),\ \sigma^-(b) \geqslant 0.
\end{array}\right\} E^{DM}
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
&k^* = \max k, \text{ subject to,} \\
&\left.\begin{array}{l}
E^{DM} \\
\sigma^+(a) + \sigma^-(a) + \sigma^+(b) + \sigma^-(b) \leqslant \overline{\sigma} + \eta(\overline{\sigma}).
\end{array}\right\} E^{DM'}
\end{aligned}
\tag{B.2}
$$

Supplementary material on the paper
Deck of Cards method for Hierarchical, Robust and Stochastic Ordinal Regression

Salvatore Corrente[a], Salvatore Greco[a], Silvano Zappalá[a]

[a] *Department of Economics and Business, University of Catania, Corso Italia, 55, 95129 Catania, Italy*

1. Tables

Table 28: Imprecise Information - Possible preference relation between regions on global level

ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
3	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**
4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
7	**1**	**1**	**1**	**0**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
9	0	1	1	0	1	0	0	0	1	1	1	0	1	1	1	1	1	1	1	1
10	0	1	1	0	0	0	0	0	1	1	0	1	1	1	1	1	1	1	1	1
11	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
12	0	1	1	0	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
13	0	1	1	0	0	0	0	0	1	0	0	1	1	1	1	1	1	1	1	1
14	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**1**	**0**	**1**	**1**	**1**
15	0	1	1	0	0	0	0	0	1	1	0	1	1	1	1	1	1	1	1	1
16	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	1	1
17	0	1	1	0	0	0	0	0	1	1	0	1	1	1	1	1	1	1	1	1
18	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
20	0	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1

Email addresses: salvatore.corrente@unict.it (Salvatore Corrente), salgreco@unict.it (Salvatore Greco), silvano.zappala@phd.unict.it (Silvano Zappalá)

Table 29: Imprecise Information - Possible preference relation between regions on Circular Economy

ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	0	1	1	0
2	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**
4	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
7	**1**	**1**	**1**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**
8	0	0	1	0	0	0	0	1	1	0	0	1	0	1	0	1	0	0	1	0
9	0	0	1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0
10	0	1	1	0	0	0	0	1	1	1	1	1	0	1	1	1	0	1	1	0
11	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**1**	**0**
12	0	0	1	0	0	0	0	1	1	0	0	1	0	0	0	0	0	0	1	0
13	0	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	0	1	1	0
14	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**0**	**1**	**0**
15	0	1	1	0	0	0	0	1	1	1	1	1	0	1	1	1	0	1	1	0
16	0	0	1	0	0	0	0	1	1	0	0	1	0	0	0	1	0	0	1	0
17	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	0
18	0	1	1	0	0	0	0	1	1	0	0	1	0	1	0	1	0	1	1	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
20	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 30: Imprecise Information - Possible preference relation between regions on Innovation-Driven Development

ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
3	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
4	0	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
5	0	1	1	0	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
6	**0**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
7	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
9	0	1	1	0	0	0	0	0	1	1	1	0	1	1	1	1	1	1	1	1
10	0	1	0	0	0	0	0	0	0	1	0	0	1	1	1	1	1	1	1	1
11	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
12	0	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
13	0	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
14	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
15	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1

Table 31: Imprecise Information - Possible preference relation between regions on Smart Specialization

ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
7	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
11	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
12	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
13	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
14	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
17	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
18	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
19	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
20	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 32: Imprecise Information - Necessary preference relation between regions on global level

ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
3	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**1**	**0**	**1**	**1**	**0**
4	1	1	1	1	0	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1
5	0	1	1	0	1	0	0	0	0	1	1	0	1	1	1	1	1	1	1	1
6	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
7	**0**	**1**	**1**	**0**	**0**	**0**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
8	0	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
9	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0	1	1	0
10	0	0	1	0	0	0	0	0	0	1	0	0	1	1	0	1	0	1	1	0
11	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**
12	0	0	1	0	0	0	0	0	1	0	0	1	0	1	0	1	0	1	1	0
13	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	0	1	1	0
14	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**1**	**1**	**0**
15	0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	1	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	1	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1

Table 33: Imprecise Information - Necessary preference relation between regions on Circular Economy

ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	0	1	0	0	0	0	1	1	1	1	1	1	1	1	1	0	1	1	0
2	0	1	1	0	0	0	0	1	1	0	0	1	0	0	0	1	0	0	1	0
3	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**
4	1	0	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	1	1	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6	**1**	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
7	**1**	**0**	**1**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**0**
8	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	0
9	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0
10	0	0	1	0	0	0	0	1	1	1	1	1	0	1	0	1	0	1	1	0
11	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	**1**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**1**	**0**
12	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0
13	0	0	1	0	0	0	0	1	1	1	1	1	1	1	1	1	0	1	1	0
14	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**0**	**1**	**0**
15	0	0	1	0	0	0	0	1	1	0	0	1	0	1	1	1	0	1	1	0
16	0	0	1	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	1	0
17	1	0	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0
18	0	0	1	0	0	0	0	1	1	0	0	1	0	1	0	1	0	1	1	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
20	1	0	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 34: Imprecise Information - Necessary preference relation between regions on Innovation-Driven Development

ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
3	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
4	0	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
5	0	1	1	0	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
6	**0**	**1**	**1**	**1**	**1**	**1**	**0**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
7	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
9	0	1	1	0	0	0	0	0	1	1	1	0	1	1	1	1	1	1	1	1
10	0	1	0	0	0	0	0	0	0	1	0	0	1	1	1	1	1	1	1	1
11	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
12	0	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
13	0	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
14	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**1**	**1**	**1**	**1**	**1**
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1

Table 35: Imprecise Information - Necessary preference relation between regions on Smart Specialization

ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**1**
4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	0	1	1	0	1	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1
6	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
7	**0**	**1**	**1**	**0**	**1**	**0**	**1**	**0**	**0**	**1**	**1**	**0**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**1**
8	0	1	1	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
9	0	1	1	0	1	0	1	0	1	1	1	0	1	1	1	1	1	1	1	1
10	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1
11	**0**	**1**	**1**	**0**	**1**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	**1**	**1**	**0**	**0**	**0**	**1**	**1**	**1**
12	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
13	0	1	1	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1
14	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**1**
15	0	1	1	0	1	0	0	0	0	1	1	0	1	1	1	1	0	1	1	1
16	0	1	1	0	1	0	0	0	0	1	0	0	1	1	0	1	0	1	1	1
17	0	1	1	0	1	0	0	0	0	1	1	0	1	1	0	1	1	1	1	1
18	0	1	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	1
19	0	1	1	0	0	0	0	0	0	1	0	0	1	1	0	0	0	1	1	1
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 36: Imprecise Information - Rank Acceptability Indices (RAIs) on global level

Rank ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	0	54.24	45.76	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	34.82	65.18	0
3	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**26.97**	**73.03**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
4	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0
6	**0**	**0**	**45.76**	**54.24**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
7	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
8	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0
11	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
12	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0	0	0	0	0	0
14	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**91.13**	**8.87**	**0**	**0**	**0**
15	0	0	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	73.03	26.97	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	65.18	34.82	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8.87	91.13	0	0	0

Table 37: Imprecise information - Rank Acceptability Indices (RAIs) on Circular Economy

Rank ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	51.64	48.36	0	0	0	0	0	0
3	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**0**
4	0	61.07	38.93	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	38.93	61.07	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
7	**0**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0
10	0	0	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0
11	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0	0	0	0
13	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0
14	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**48.36**	**51.64**	**0**	**0**	**0**	**0**	**0**	**0**
15	0	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0	0	0	0	0
17	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100
20	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 38: Imprecise Information - Rank Acceptability Indices (RAIs) on Innovation-Driven Development

Rank ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0.03	99.97	0	0	0	0	0
3	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
4	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	**0**	**0**	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
7	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
8	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0
11	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
12	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0
14	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**89.41**	**10.59**	**0**	**0**	**0**	**0**	**0**	**0**
15	0	0	0	0	0	0	0	0	0	0	0	0	10.59	89.38	0.03	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0	0

Table 39: Imprecise Information - Rank Acceptability Indices (RAIs) on Smart Specialization

Rank / ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	9.81	90.19
3	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**0**
4	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	22.39	77.61	0	0	0	0	0	0
6	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
7	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
8	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0	0	0	0
11	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
12	0	100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0	0	0	0	0
14	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**0**	**0**
15	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	77.61	22.39	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	90.19	9.81

Table 40: Imprecise Information - Pairwise Winning Indices (PWIs) on global level

ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	100	100	0	100	54.24	100	0	100	100	100	100	100	100	100	100	100	100	100	100
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	34.82	100	0
3	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**100**	**0**	**100**	**26.97**	**100**	**100**	**100**
4	100	100	100	0	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
5	0	100	100	0	0	0	0	0	100	100	100	0	100	100	100	100	100	100	100	100
6	**45.76**	**100**	**100**	**0**	**100**	**0**	**100**	**0**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
7	**0**	**100**	**100**	**0**	**100**	**0**	**0**	**0**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
8	100	100	100	0	100	100	100	0	100	100	100	100	100	100	100	100	100	100	100	100
9	0	100	100	0	0	0	0	0	0	100	100	0	100	100	100	100	100	100	100	100
10	0	100	100	0	0	0	0	0	0	0	0	0	100	100	0	100	100	100	100	100
11	**0**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
12	0	100	100	0	100	0	0	0	100	100	100	0	100	100	100	100	100	100	100	100
13	0	100	0	0	0	0	0	0	0	0	0	0	0	100	0	100	0	100	100	100
14	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**100**	**91.13**
15	0	100	100	0	0	0	0	0	0	100	0	0	100	100	0	100	100	100	100	100
16	0	100	0	0	0	0	0	0	0	0	0	0	0	100	0	0	0	100	100	100
17	0	100	73.03	0	0	0	0	0	0	0	0	0	100	100	0	100	0	100	100	100
18	0	65.18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	100	0	0	0	0	0	0	0	0	0	0	0	8.87	0	0	0	100	100	0

Table 41: Imprecise Information - Pairwise Winning Indices (PWIs) on Circular Economy

ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	100	100	0	0	0	0	100	100	100	100	100	100	100	100	100	0	100	100	0
2	0	0	100	0	0	0	0	100	100	0	0	100	0	51.64	0	100	0	0	100	0
3	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**0**
4	100	100	100	0	61.07	0	100	100	100	100	100	100	100	100	100	100	100	100	100	100
5	100	100	100	38.93	0	0	100	100	100	100	100	100	100	100	100	100	100	100	100	100
6	**100**	**100**	**100**	**100**	**100**	**0**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
7	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**0**	**100**	**100**	**0**
8	0	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	100	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0
10	0	100	100	0	0	0	0	100	100	0	100	100	0	100	0	100	0	100	100	0
11	**0**	**100**	**100**	**0**	**0**	**0**	**0**	**100**	**100**	**0**	**0**	**100**	**0**	**100**	**0**	**100**	**0**	**100**	**100**	**0**
12	0	0	100	0	0	0	0	100	100	0	0	0	0	0	0	0	0	0	100	0
13	0	100	100	0	0	0	0	100	100	100	100	100	0	100	100	100	0	100	100	0
14	**0**	**48.36**	**100**	**0**	**0**	**0**	**0**	**100**	**100**	**0**	**0**	**100**	**0**	**0**	**0**	**100**	**0**	**0**	**100**	**0**
15	0	100	100	0	0	0	0	100	100	100	100	100	0	100	0	100	0	100	100	0
16	0	0	100	0	0	0	0	100	100	0	0	100	0	0	0	0	0	0	100	0
17	100	100	100	0	0	0	100	100	100	100	100	100	100	100	100	100	0	100	100	0
18	0	100	100	0	0	0	0	100	100	0	0	100	0	100	0	100	0	0	100	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	100	100	100	0	0	0	100	100	100	100	100	100	100	100	100	100	100	100	100	0

Table 42: Imprecise Information - Pairwise Winning Indices (PWIs) on Innovation-Driven Development

ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	100	100	100	100	100	0	0	100	100	100	100	100	100	100	100	100	100	100	100
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.03	100	100	100	100	100
3	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
4	0	100	100	0	100	0	0	0	100	100	100	100	100	100	100	100	100	100	100	100
5	0	100	100	0	0	0	0	0	100	100	100	100	100	100	100	100	100	100	100	100
6	**0**	**100**	**100**	**100**	**100**	**0**	**0**	**0**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
7	**100**	**100**	**100**	**100**	**100**	**100**	**0**	**0**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
8	100	100	100	100	100	100	100	0	100	100	100	100	100	100	100	100	100	100	100	100
9	0	100	100	0	0	0	0	0	0	100	100	0	100	100	100	100	100	100	100	100
10	0	100	0	0	0	0	0	0	0	0	0	0	100	100	100	100	100	100	100	100
11	**0**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
12	0	100	100	0	0	0	0	0	100	100	100	0	100	100	100	100	100	100	100	100
13	0	100	0	0	0	0	0	0	0	0	0	0	0	100	100	100	100	100	100	100
14	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**89.41**	**100**	**100**	**100**	**100**	**100**
15	0	99.97	0	0	0	0	0	0	0	0	0	0	0	10.59	0	100	100	100	100	100
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	100	100	100
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	100	100
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	100	0

Table 43: Imprecise Information - Pairwise Winning Indices (PWIs) on Smart Specialization

ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	100	100	0	100	100	100	100	100	100	100	0	100	100	100	100	100	100	100	100
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	9.81
3	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**100**
4	100	100	100	0	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
5	0	100	100	0	0	0	0	0	0	100	0	0	100	100	0	0	0	22.39	0	100
6	**0**	**100**	**100**	**0**	**100**	**0**	**100**	**0**	**100**	**100**	**100**	**0**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
7	**0**	**100**	**100**	**0**	**100**	**0**	**0**	**0**	**0**	**100**	**100**	**0**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
8	0	100	100	0	100	100	100	0	100	100	100	0	100	100	100	100	100	100	100	100
9	0	100	100	0	100	0	100	0	0	100	100	0	100	100	100	100	100	100	100	100
10	0	100	100	0	0	0	0	0	0	0	0	0	0	100	0	0	0	0	0	100
11	**0**	**100**	**100**	**0**	**100**	**0**	**0**	**0**	**0**	**100**	**0**	**0**	**100**	**100**	**0**	**0**	**0**	**100**	**100**	**100**
12	100	100	100	0	100	100	100	100	100	100	100	0	100	100	100	100	100	100	100	100
13	0	100	100	0	0	0	0	0	0	100	0	0	0	100	0	0	0	0	0	100
14	**0**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**100**
15	0	100	100	0	100	0	0	0	0	100	100	0	100	100	0	100	100	100	100	100
16	0	100	100	0	100	0	0	0	0	100	100	0	100	100	0	0	0	100	100	100
17	0	100	100	0	100	0	0	0	0	100	100	0	100	100	0	100	0	100	100	100
18	0	100	100	0	77.61	0	0	0	0	100	0	0	100	100	0	0	0	0	0	100
19	0	100	100	0	100	0	0	0	0	100	0	0	100	100	0	0	0	100	0	100
20	0	90.19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 44: Imprecise Information - Parameters of the approximated barycenter

$w_{(\mathbf{1},\mathbf{1})}$	$w_{(\mathbf{1},\mathbf{2})}$	$w_{(\mathbf{1},\mathbf{3})}$	$w_{(\mathbf{2},\mathbf{1})}$	$w_{(\mathbf{2},\mathbf{2})}$	$w_{(\mathbf{2},\mathbf{3})}$	$w_{(\mathbf{3},\mathbf{1})}$	$w_{(\mathbf{3},\mathbf{2})}$	$w_{(\mathbf{3},\mathbf{3})}$
0.1132	0.1413	0.0055	0.3909	0.0432	0.0045	0.0213	0.1470	0.1333

Table 45: Imprecise Information - Ranking obtained using the approximation of the barycenter at the global and partial levels

(a) Global level - $g_{(\mathbf{0})}$

Rank	Region	ID	$U(a_j)$
1	Lombardy	4	0.68507
2	Emilia-Romagna	8	0.67725
3	**Piedmont**	**1**	**0.63784**
4	Veneto	6	0.63754
5	Friuli-Venezia Giulia	7	0.61808
6	**Lazio**	**12**	**0.58197**
7	**Trentino-South Tyrol**	**5**	**0.56066**
8	Tuscany	9	0.53991
9	Marche	11	0.51384
10	Campania	15	0.47194
11	**Umbria**	**10**	**0.45911**
12	Basilicata	17	0.44823
13	Liguria	3	0.44756
14	**Abruzzo**	**13**	**0.43893**
15	Apulia	16	0.40177
16	Molise	14	0.3919
17	Sardinia	20	0.38905
18	Calabria	18	0.37864
19	Aosta Valley	2	0.3774
20	Sicily	19	0.33549

(b) Circular Economy $g_{(\mathbf{1})}$

Rank	Region	ID	$U(a_j)$
1	Veneto	6	0.16497
2	Lombardy	4	0.16023
3	**Trentino-South Tyrol**	**5**	**0.15999**
4	Sardinia	20	0.15675
5	Basilicata	17	0.15154
6	**Friuli-Venezia Giulia**	**7**	**0.14851**
7	**Piedmont**	**1**	**0.14144**
8	Abruzzo	13	0.14075
9	Campania	15	0.13838
10	Umbria	10	0.13564
11	**Marche**	**11**	**0.13179**
12	Calabria	18	0.12818
13	Aosta Valley	2	0.11782
14	**Molise**	**14**	**0.11746**
15	Apulia	16	0.11098
16	Lazio	12	0.10812
17	Liguria	3	0.10636
18	Emilia-Romagna	8	0.10318
19	Tuscany	9	0.097505
20	Sicily	19	0.079469

(c) Innovation-Driven Development $g_{(\mathbf{2})}$

Rank	Region	ID	$U(a_j)$
1	Emilia-Romagna	8	0.37848
2	Friuli-Venezia Giulia	7	0.30325
3	**Piedmont**	**1**	**0.297**
4	Veneto	6	0.28597
5	Lombardy	4	0.28441
6	**Trentino-South Tyrol**	**5**	**0.27447**
7	**Lazio**	**12**	**0.27139**
8	Tuscany	9	0.26303
9	Marche	11	0.2378
10	Liguria	3	0.22915
11	**Umbria**	**10**	**0.20654**
12	Abruzzo	13	0.17597
13	Molise	14	0.1706
14	**Campania**	**15**	**0.16966**
15	Aosta Valley	2	0.16893
16	Apulia	16	0.14076
17	Basilicata	17	0.14014
18	Sardinia	20	0.13996
19	Sicily	19	0.1242
20	Calabria	18	0.12297

(d) Smart Specialization $g_{(\mathbf{3})}$

Rank	Region	ID	$U(a_j)$
1	Lombardy	4	0.24043
2	Lazio	12	0.20246
3	**Piedmont**	**1**	**0.1994**
4	Emilia-Romagna	8	0.19559
5	Veneto	6	0.18659
6	**Tuscany**	**9**	**0.17937**
7	**Friuli-Venezia Giulia**	**7**	**0.16632**
8	Campania	15	0.1639
9	Basilicata	17	0.15654
10	Apulia	16	0.15002
11	**Marche**	**11**	**0.14424**
12	Sicily	19	0.13182
13	Calabria	18	0.1275
14	**Trentino-South Tyrol**	**5**	**0.12621**
15	Abruzzo	13	0.12221
16	Umbria	10	0.11693
17	Liguria	3	0.11205
18	Molise	14	0.10384
19	Sardinia	20	0.09234
20	Aosta Valley	2	0.090645