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Multilateral market power in input-output networks[*,†]

Matteo Bizzarri [‡]

June 2025

Abstract

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Introduction

Who has stronger market power in a production network? This question is connected with fundamental questions such as the welfare evaluation of mergers, the quantification of distortions due to market power, the response of the economy to shocks. In this paper, I argue that, to properly answer the question, it is important to use models were firms are allowed to have *multilateral market power*, that is they can potentially affect prices both on input and output markets.

Recent papers find input market power to be sizable, both domestically (Morlacco (2019), Dhyne et al. (2022)), and in international trade (Alviarez et al., 2023). However, with the exceptions noted in the literature, most customarily used models of firm-to-firm trade impose the simplifying assumption that firms are price-takers on the input markets. In this paper, I study a strategic non-cooperative model of large firms interacting in an input-output network consisting of many specific supply-customer relationships. Firms trade using a double auction for each good, as in models of the financial market (Malamud and Rostek, 2017).

The main contributions of the paper are two. First, I show that with Leontief technology the game in schedules has a unique linear equilibrium. Then, I explore the connections between the network and the equilibrium markups and markdowns: there is an appropriate weighted network connecting the goods of the economy, the *goods network*, such that the markups and markdowns can be seen as a measure of centrality with respect to this network. Second, I explore what are the implications of multilateral market power. To do so, I compare the equilibrium of the benchmark model with a modified model where firms are price-takers on inputs, and a version where firms take as given all the prices of markets they are not directly involved in. When firms take some prices as given, the distortions due to market power are lower: the equilibrium price impacts are lower, and the final price is also lower.

Formally, firms have a set of input goods, and produce each an output: some outputs are the input of other firms and these trade relationships, or *input-output links*, are exogenous. Firms play a simultaneous game in which the available actions are supply and demand schedules, relating quantities of the traded goods to prices: as in a uniform-price double auction, the realized price on every trade relationship is the one where demand and supply cross. The classic metaphor for the price-taking general equilibrium behavior is that a "walrasian" auctioneer proposes prices and collects supply and demand "bids", until all markets clear. The approach followed in this paper takes this metaphor one step further, applying it to non-infinitesimal firms. The auctioneer acts as a market maker in financial markets, collecting firms' conditional schedules. Firms, being non-infinitesimal, fully internalize the mechanism and submit their schedules to affect prices in their favor.[1]

Section 2 introduces the model. The technological assumption is that intermediate inputs are perfect complements. The assumption on labor generalizes slightly the Leontief functional form, allowing the quantity of labor to depend quadratically on the output quantity. I do so for two reasons: first, it simplifies the conditions under which an equilibrium exists; second, this technology recovers as a special case the standard quadratic cost function often used in models with no vertical connections (e.g. Klemperer and Meyer (1989), Pellegrino (2025)).

In the main text I assume that firms directly choose linear schedules that, under the technology constraint, boils down to assuming that firms choose a single number, representing the slope of the supply schedule. The equilibrium cannot be computed analytically, but the game so defined has many useful properties that make it tractable: it is a supermodular potential game. Theorem 1 shows that this game has a unique equilibrium for very general networks, provided any good is traded by at least three agents (which may be two firms and the consumers). This is a classic condition for the existence of the linear equilibrium in models of competition in supply functions (Malamud and Rostek, 2017). In the Appendix **??** I show that the same profile of linear schedules indeed arises in equilibrium[2] in a model where there are uncertain cost parameters of firm's technology, and schedules have to be chosen before the realizations.

[1]This competition in schedules is meant not as a literal description of the workings of the market (although they are in some cases, e.g. the electricity or financial markets), but as an abstraction of a bargaining procedure, parsimonious but powerful enough for the complexity of the problem.

[2]By which I mean that the best response over *all* feasible schedules is linear.

Section 4 define the goods network, and characterize the connection between market power and the network. In such a network, the nodes are the goods, and two goods are linked if some firms trade both goods. The strength of a link. The equilibrium prices are proportional to the centrality in such a network, as is common in network models. What is more interesting, and specifically connected to market power, is that also the price impacts have a network interpretation: for example, the price impact of firm i on its output is proportional to the (weighted) number of cycles in the goods network, excluding firm i. This is because the network affects the price impact via the pass-through of prices: the number of cycles measures the strength of the pass-through effect.

In the special case of the Supply chain with layers, all the effects can be precisely characterized: we obtain that, in the homogeneous situation where all layers have the same size and number of firms, the markup (and the price impact on the output) is larger for the more upstream firm (the farther from the consumer), while the markdown is larger for the more downstream firm (the closer to the consumer).

Section 5 generalizes the model, allowing general price impact functions satisfying the technical condition that they must be decreasing (in the positive semidefinite sense) in the slopes of the schedules. This defines a "Generalized SDFE", that encompasses as special cases several standard models: from the classic Cournot and Bertrand oligopoly (without input-output dimension) to the sequential monopoly a la Spengler (1950). Moreover, Theorem 3 shows that if one takes the benchmark SDFE and imposes the assumptions that (i) firms take *input* prices as given (which I call *unilateral market power*) and (ii) firms take as given all prices of markets where they are not directly involved in (*local market power*, also these two models are Generalized SDFE for the proper choice of the price impact function.

Theorems 2 and 3, together, constitute the main result of the paper: the description of the implications of multilateral market power in constrast with, in the above terminology, unilateral and local market power. Theorem 2 exploits that the game still is a supermodular game to prove existence, and to show the main comparative statics: if the price impact function is larger (in the positive semidefinite sense), the equilibrium slopes are smaller. Theorem 3 shows that both with unilateral and local market power the price impacts are smaller. As a consequence, the final price is smaller under these two assumptions.

The intuition is as follows. If a firm does not internalize some reactions in the network, this amounts to that firm perceiving a larger elasticity of demand and supply and, as a consequence, being able to charge smaller markups and markdowns. This is because, in the S&D equilibrium, the elasticity of demand depends on the schedules chosen by directly connected firms, but also *indirectly* connected firms. The reason is that, in equilibrium, a change in a price triggers a change in all other prices of connected firms: failing to account for some of these pass-through effects means firms perceive a different elasticity of demand.[3]

Multilateral market power does not only have a global effect, but also changes the balance of market power among firms. Proposition 2 illustrates this effect for the supply chain with layers. When market power is multilateral, as described above, all the layers are symmetric. Instead, with unilateral market power, the upstream layers have larger markup and profit than the downstream layers. So, the amount of surplus extraction predicted by the model is very different.

These considerations suggest that models that impose restrictions on which prices a firm can affect are not innocuous, and must be taken with care, especially if these assumptions are just a simplifying modeling device, rather than coming from specific timing of the market mechanism. This is typically true of models of general input-output networks that connect many firms that are very heterogeneous in terms of the nature of their processes and products, and so very specific assumptions on which prices firms can or cannot control (or specific timing assumptions) are harder to justify.

Related literature

This paper contributes to three lines of literature: the literature on competition in supply and demand functions, the literature on production

[3]The literature on outsourcing and endogenous supply chains provides evidence that firms are aware of their supply chain and take its structure and their position in it into account in their decisions, see e.g. Berlingieri et al. (2020), Alfaro et al. (2019).

networks or networked markets, and the literature on general equilibrium oligopoly.

My contribution to the literature on competition on supply and demand functions is to introduce the technique to the modeling of general equilibrium oligopoly, in particular with firm-to-firm trade. The literature has studied the situation where the demand firms receive comes from a network structure with a large dimension of uncertainty, in Wilson (2008), Holmberg and Philpott (2018), Ruddell (2018a), Ruddell (2018b), but their firms only supply to a node in the network, do not trade among themselves. Firm-to-firm trade is studied in a bilateral setting in Weretka (2011) and Hendricks and McAfee (2010), always constraining the schedules to a parametric functional form. In the finance literature the model is used to study simultaneous demand and supply of heterogeneous assets: Malamud and Rostek (2017) show how the strategic complementarity property extends to the network setting, and characterizes an equilibrium in a general network. The model has a different purpose (studying centralization in financial markets) and also two important technical differences: in my paper the functional form is different, because the Leontief technology gives a different best reply equation: the difference is important, because it is crucial in obtaining uniqueness of the equilibrium. Moreover, I study the Generalized SDFE version with general price impacts. Rostek and Yoon (2021a), Rostek and Yoon (2021b) and Rostek and Weretka (2012) also analyze similar models, and share the same differences with my work. Ausubel et al. (2014) and Woodward (2021) study uniform-price (among other) auctions in the context of centralized auctions. Vives (2011) studies market power arising from asymmetric information, rather than network position.

My contribution to the production networks literature is to provide a model of competition in an input-output network in which all firms have market power on both input and output markets, and are fully strategic internalizing their position in the supply chain. Many models explicitly assume that firms have power to decide/affect prices only on one side of the market. To this class belong the workhorse sequential oligopoly games in Salinger (1988), Ordover et al. (1990), Hart et al. (1990).[4] In another class of models authors assume that output prices are equal to the marginal cost

[4]And used in classic textbook treatments, such as Tirole (1988).

times a markup. The concept of the marginal cost itself implicitly implies price-taking in the input market: indeed, it arises from the price-taking cost minimization problem of the firm. Hence, it is implicitly assuming unilateral market power. To this category belong Grassi (2017), Bernard et al. (2022), Baqaee (2018), Baqaee and Farhi (2019), Baqaee and Farhi (2020), Magerman et al. (2020), Dhyne et al. (2022), Huneeus et al. (2021), Arkolakis et al. (2021), Pasten et al. (2020), Pellegrino (2025). A third class of models are those where vertically connected firms share surplus via some form of Nash bargaining. Toxvaerd (2024) reviews the recent work in the area, in the context of a vertical chain. Acemoglu and Tahbaz-Salehi (2025) and Alviarez et al. (2023) apply this idea to general networks. My results complements theirs, providing a model that does not rely on the choice of exogenously specified bargaining weights.[5] More in general, many models of networked markets have studied the network defined by the demand: Galeotti et al. (2024), Pellegrino (2025), Bimpikis et al. (2019), but they do not focus on input-output connections.

Except for Acemoglu and Tahbaz-Salehi (2025), all these papers feature also the implicit or explicit assumption that firms do not internalize the effect of their decisions on sectors/firms further downstream beside the direct customers. Sometimes this is a consequence of the assumption of a continuum of firms in each sector (and so sector-level aggregates are taken as given by every individual firm),[6] other times it is explicitly assumed.[7]

I contribute to the literature on general equilibrium with market power by providing a fully strategic model of the production side with endogenous market power and firm-to-firm trade; furthermore, the game does not depend on price normalization, and can incorporate general assumptions on owner's preferences as in Azar and Vives (2021). In the recent literature on "general oligopolistic competition" (Azar and Vives (2021), Azar and Vives (2018) and Ederer and Pellegrino (2022)) do not consider firm-to-firm trade.

[5] The papers also differ from mine in other dimensions: Alviarez et al. (2023) study buyer-seller, rather than input-output connections; Acemoglu and Tahbaz-Salehi (2025) is a model of endogenous exit: in the benchmark with no exit, the equilibrium is efficient, unlike in my model.

[6] This is the case in, e.g. Baqaee (2018) and various others listed in the literature.

[7] E.g., in Grassi (2017), Dhyne et al. (2022).

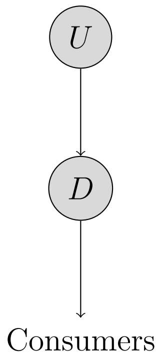


Figure 1: A simple supply chain with two layers: U and D.

The rest of the paper is organized as follows. Section 2 defines the benchmark model, the Supply and Demand Function Equilbrium (SDFE). Section 3 describes the solution and the existence theorem. Section 4 shows the characterization of markups and the connection with the goods network. Section 5 introduces the Generalized SDFE, and explores the effect of multilateral market power. Section 6 concludes. The proofs are in the Appendix.

1 A simple example

In this section we illustrate the model and the main take-aways in a the simplest network where the concept of multi-lateral market power is non-trivial: a supply chain consisting of one intermediate producer, U, and a final good producer D. This is represented below in Figure 1. The intermediate good producer U produces good U using only labor, and sell it to the final producer D. In turn, the final producer D uses good U to produce the final output D. The consumers consume both goods U and D.

Consumers have linear demands for both goods that, for simplicity, is linear with slope normalized to 1, and the goods are neither substitutes nor complements:

$$D_{c,D}(p_D) = A - p_D$$
$$D_{c,U}(p_U) = A - p_U$$

The firms have linear technology, producing one unit of output for any unit of input: $F_U(q_U) = q_U$ and $F_D(q_D) = q_D$. So, firms profits, when the realized prices are p_D, p_U, are:

$$\pi_U = p_U q_U \tag{1}$$

$$\pi_D = (p_D - p_U) q_D \tag{2}$$

The firms play a simultaneous game in which the strategic variables are the (slopes of the) linear schedules connecting prices and quantities. Formally:

1. firm U submits a function $S_U(p_U) = B_U p_U$, where B_U is any positive real number;

2. firm D submits a function

$$S_D(p_U, p_D) = B_D(p_D - p_U)$$

   indicating both its supply of output, and its demand for the input, where B_D is, again, any positive real number.

Whichever choice of the firms, the prices p_U, p_D and quantities q_U, q_D must satisfy the market clearing conditions:

$$q_D = A - p_D = B_D(p_D - p_U) \tag{3}$$

$$q_U = A - p_U + B_D(p_D - p_U) = B_U p_U \tag{4}$$

A wide variety of allocations realize for different choices of schedules. For example, perfect competition is the special case in which B_U and B_D go to infinity. Indeed, it turns out that in this example welfare is increasing in both slopes. We want to extract predictions on firms behavior by looking for a Nash equilibrium of the game in schedules, in which firms aim to maximize profits.

Focus on firm U. For each fixed quantity of output q_U, we can solve the above system for the inverse demand:

$$p_{U,U}(q_U) = \left(1 + \frac{B_D}{B_D + 1}\right)^{-1} (A - q_U)$$

and similarly for the inverse demands faced by firm D, that we call $p_{D,U}(q_D)$ and $p_{D,D}(q_D)$. When taking the FOC for firm U, we get:

$$\frac{\partial}{\partial B_U}\pi_U = \frac{\partial q_U}{\partial B_U}\left(p_U + q_U\frac{\partial p_{U,U}}{\partial q_U}\right) = 0$$

From the market clearing conditions it is easy to conclude that $\dfrac{\partial q_U}{\partial B_U} > 0$, and so the FOC are equivalent to: $p_U + q_U\dfrac{\partial p_{U,U}}{\partial q_U} = 0$. Doing the analogue for firm D, we obtain the equilibrium equations:

$$p_U + q_U\frac{\partial p_{U,U}}{\partial q_U} = 0 \tag{5a}$$

$$p_D - p_U + q_D\left(\frac{\partial p_{D,D}}{\partial q_D} - \frac{\partial p_{D,U}}{\partial q_D}\right) = 0 \tag{5b}$$

Since schedules are linear, the derivatives are just constants: so, it is immediate to write the best response schedules as:

$$\begin{aligned} S_U(p_U) &= \left(-\frac{\partial p_{U,U}}{\partial q_U}\right)^{-1} p_U \\ S_D(p_D, p_U) &= \left(\frac{\partial p_{D,D}}{\partial q_D} - \frac{\partial p_{D,U}}{\partial q_D}\right)^{-1} (p_D - p_U) \end{aligned}$$

So, the slopes B_D^*, B_U^* that constitute an equilibrium of the game must be equal to the slope of the above functions, and satisfy:

$$\begin{aligned} B_U^* &= \left(-\frac{\partial p_{U,U}}{\partial q_U}\right)^{-1} = 1 + \frac{B_D^*}{B_D^* + 1} \\ B_D^* &= \left(\frac{\partial p_{D,D}}{\partial q_D} - \frac{\partial p_{D,U}}{\partial q_D}\right)^{-1} = \left(1 + \frac{1}{B_U^*}\right)^{-1}. \end{aligned} \tag{6}$$

The expression highlights the role of the price impacts, and in particular, the fact that firm D has price impact on both the input and the output market. The equations can be solved analytically, and it can be checked that the solution is: $B_D^* = 1/\sqrt{2}$, $B_U^* = \sqrt{2}$.

What happens if firm D is a price-taker on the input market? In that case the choice of D does not effect the input price, $\dfrac{\partial p_{D,U}}{\partial q_D} = 0$, and so the equilibrium equations (6) become:

$$B_U^{**} = 1 + \frac{B_D^{**}}{B_D^{**} + 1}$$
$$B_D^{**} = 1.$$

Moreover, this solution is the same we would get solving the model as a standard sequential monopoly, as shown in Section REFERENCE. The solution in this case is $B_D^{**} = 1$, $B_U^{**} = 3/2$. They are both higher than in the case of multilateral market power. So, we can immediately conclude that consumer welfare is higher in this case.

In the rest of the paper, we explore how this insight generalizes to arbitrary networks, and we illustrate examples how taking multilateral market power into account can change the model conclusions on the welfare impact of mergers and diffusion of shocks.

So, in a sense, both firms set their "optimal price" on the U, D link. This seems a contradiction, since sellers would want to raise p_U while buyers would want to decrease it. The tension is resolved by the fact that here firms "implement" a price by modifying the slope of their schedule that, in turn, *changes other firms' incentives to raise prices.* The situation is represented graphically in Figure 2: firm U faces a residual demand $D_U^r(p_U)$, that is the blue line in the graph, depending on the slope of consumers and the slope chosen by D. This residual demand induces a profit as a function of the price p_U. Firm U wants to charge p_U^*, the monopoly price for this residual demand, and so sets a slope that achieves that price: this is the red line in Figure 2a. But, in doing so, it affects the slope of the *residual supply* that firm D faces. As a consequence, firm D changes their choice of schedule, changing the transaction price to $(p_U^*)_2$, the optimal *monopsony* price for firm D. This, in turns, leads to a new residual demand and a new profit function for firm U (as in Figure 2b): as a consequence, the previous optimal price p_U^* is not optimal anymore, and firm U adjusts its slope again. This adjustment process continues until the slopes are such that the optimal price sellers want to charge is equal to the optimal price for the buyers.

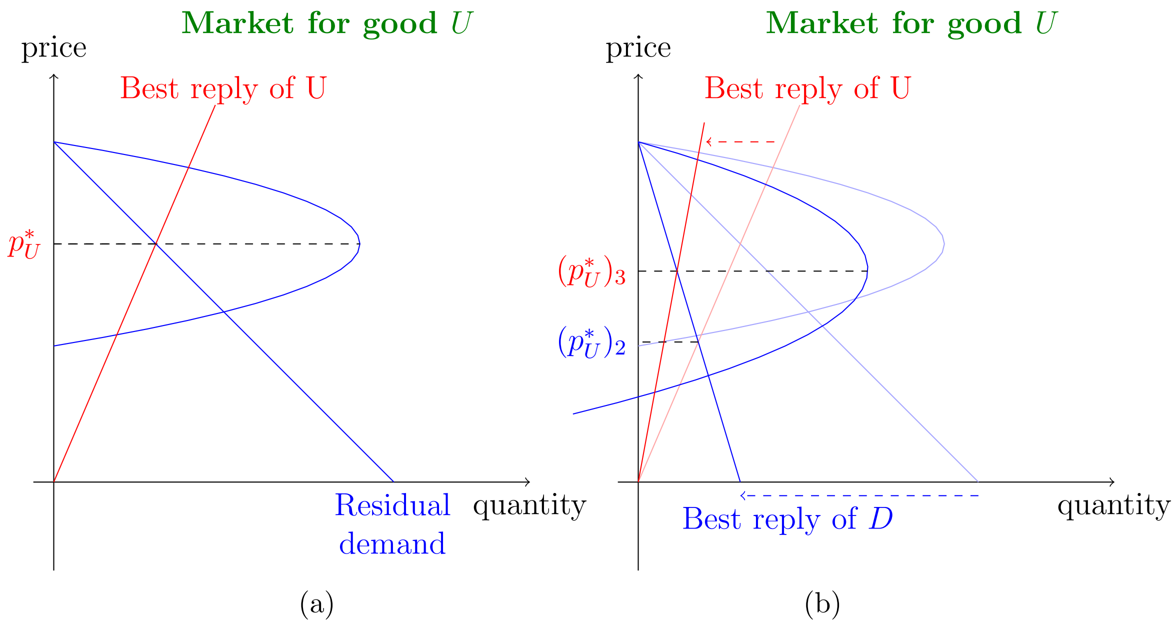


Figure 2: Graphical representation of the choice of schedule by firm U. On the left (a): the best reply for firm U to the residual demand given by the blue line. On the right (b): the optimal choice of firm U leads other firms to adjust, modifying firm U residual demand and optimal price: so firm U further adjusts its best reply.

2 The model

In this section I introduce the primitives of the model, that are the firms and their technology, the input-output network, and the utility of the consumer, and then define the game played by the firms.

2.1 Setting

Firms and Production Network There are n firms and m goods: their sets are respectively denoted $\mathcal{N}$ and $\mathcal{M}$. Each good might be produced by more firms, but each firm produces only one good. Each firm produces using labor and a set of inputs produced by other firms, which I denote as $\mathcal{N}_i^{in}$. Denote the set of all goods traded by firm i as $\mathcal{N}_i = \mathcal{N}_i^{out} \cup \{i\}$. The consumers' utility depends directly on a subset of goods, denoted $\mathcal{C} \subseteq \mathcal{M}$. Firms, goods and the connections defined above define a directed bipartite graph $\mathcal{G} = (\mathcal{N}, \mathcal{M}, E)$, where $E \subseteq (\mathcal{N} \cup \mathcal{M})^2$ is the set of existing connections. I refer to $\mathcal{G}$ as the *input output network* of this economy. If

$(i, g) \in E$ means that firm i produces good g, and (g, i) means that firm i needs good g for production. For brevity, I write $i \to g$ in the former case, and $g \to i$ in the latter .

Notation I denote $d_i^{in} = |\mathcal{N}_i^{in}|$ the *in-degree* (number of intermediate inputs) of firm i, excluding labor, and $d_i = d_i^{in} + 1$ the total degree (number of goods traded). We use the wage as the numeraire: the price of good g in labor terms is denoted p_g. Bold symbols are used to denote vectors: $\boldsymbol{p}$ is the vector of all prices (always in labor terms), while $\boldsymbol{p}_i^{in} = ((p_g)_{g \in \mathcal{N}_i^{in}})$ are the prices of all input goods of firm i, and similarly p_i^{out} is the price of the output, so that $\boldsymbol{p}_i' = (p_i^{out}, (\boldsymbol{p}_i^{in})')$. Similarly, $\boldsymbol{p}_c = (p_g)_{g \in \mathcal{C}}$ is the vector of prices of goods consumed by the consumer.

For quantities, it is understood that positive quantities represent outputs and negative quantities represent inputs. So, the vector of input and output quantities traded by firm i is $\boldsymbol{q}_i = (q_i, -\boldsymbol{q}_i^{in})$, where $\boldsymbol{q}_i^{in} = (q_{ig})_{g \to i}$ is the vector of input quantities. The quantity of labor used by firm i is $\tilde{\ell}_i$. For a firm choosing quantities $\boldsymbol{q}_i, \tilde{\ell}_i$, the profit is:

$$\Pi_i = \boldsymbol{p}_i' \boldsymbol{q}_i - \tilde{\ell}_i$$

where note that the wage is 1, because all prices are expressed in labor terms.

If M is a matrix, $[M]_{-i}$ denotes the same matrix to which the rows and columns relative to input and output goods of good i have been removed. If $\boldsymbol{b}$ is a vector, $\boldsymbol{b}_{-i}$ denotes the same vector to which element i has been removed.

Consumers The utility function of the consumers is quadratic in consumption and (quasi-)linear in the disutility of labor L:

$$U(\boldsymbol{c}, L) = \boldsymbol{A}' B_c^{-1} \boldsymbol{c} - \frac{1}{2} \boldsymbol{c}' B_c^{-1} \boldsymbol{c} - L \tag{7}$$

where $\boldsymbol{c} = (c_g)_{g \in \mathcal{M}}$ is the vector of quantities consumed, $\boldsymbol{A}_c$ is a vector, and B_c is a symmetric positive definite matrix. This means that the consumer demand has the form: $D_c = \boldsymbol{A} - B_c \boldsymbol{p}_c$.

Technology Intermediate inputs are perfect complements, so that to produce a quantity of output q_i firm i needs $f_{ih}q_i$ units of input h are needed. We denote $F \in \mathbb{R}_+^{m \times m}$ the matrix that collects the f_{gh}. For the technology to be viable, we adopt the standard assumption that there must exist a positive quantity vector $\boldsymbol{q}$ such that $q_i > \sum_h f_{hi}q_h$.We slightly generalize the Leontief technology to allow decreasing returns in labor, so that: to produce q_g^{out} units of output the firm needs $\tilde{\ell}_g = f_{g,L}q_g^{out} + \dfrac{1}{2k_g}(q_g^{out})^2$ labor units. We do this because, as illustrated below, the decreasing returns in labor facilitate the existence of a non-trivial equilibrium. So, if $k_i \to \infty$, the technology becomes the standard Leontief one; if $k_i < \infty$, it comes from a variation of the Leontief production function, illustrated in REFERENCE. We can write the technology constraints of firm i as:

$$\begin{aligned} q_{ij} &= f_{ij}q_i \quad \forall j \in \mathcal{N}_i^{in} \\ \tilde{\ell}_i &= f_{i,L}q_i^{out} + \frac{1}{2k_i}(q_i^{out})^2 \end{aligned} \tag{8}$$

It is going to be convenient to define the vector $\boldsymbol{v}_i = (1, -f_{i1}, \ldots, -f_{1N})$.

2.2 The game

Schedules The competition among firms take the form of a game in which firms compete choosing a supply function for the output, and demand functions for intermediate inputs and labor, respecting the technology constraint (8). The players of the game are the firms: $i = 1, \ldots, N$, and the actions available to each firm i are *linear schedules*, one for the output $\mathcal{S}_i^{out}$, and others for intermediate inputs $\mathcal{S}_i^{in}$, and labor $\mathcal{S}_{\ell,i} \in \mathbb{R}_+$. Denote the schedule of intermediate input trades of firm i as: $\mathcal{S}_i = (\mathcal{S}_i^{out}, -\mathcal{S}_i^{in})$.[8]The assumption of linearity means that there exist a matrix of coefficients $B_i \in \mathbb{R}^{d_i \times d_i}$ and a vector $B_{i,f} \in \mathbb{R}^{d_i}$, such that the schedule is linear:

$$\mathcal{S}_i(\boldsymbol{p}_i) = B_i\boldsymbol{p}_i - B_{i,f}f_{i,L}$$

[8]We denote the quantities as $\boldsymbol{q}_i$ when they are simply variables, with $\mathcal{S}_i$ when they are explicit functions of prices.

The technology constraints (8) imply that the supply function $\mathcal{S}_i^{out}$ determines the whole input schedule, as inputs are bought in constant proportion, so that: $\mathcal{S}_i = \mathcal{S}_i^{out}\boldsymbol{v}_i$. The schedule $\mathcal{S}_i^{out}$ is linear, and it turns out that it is without loss to focus on $\mathcal{S}_i^{out} = B_i(\boldsymbol{v}_i'\boldsymbol{p}_i - f_{i,L})$ for some $B_i \in \mathbb{R}_+$, as proven in Theorem 1.

In the Supplementary Appendix, it is shown that the linear equilibrium studied in the main text remains an equilibrium (and in some case it can be proven to be unique) also when firms are not constrained to choose a linear schedule: the linear schedule is the unconstrained best reply among all possible schedules.

Prices The market prices are, by assumption, those satisfying the market clearing equations. Since the demand derived by (7) satisfies Walras's law, it is standard that one of the market clearing conditions is redundant: we leave out the labor market clearing equation $\sum_i \tilde{\ell}_i(\boldsymbol{p}_i) = L(\boldsymbol{p}_c)$, and write the market clearing system as:

$$\sum_{i\,:\,g\in\mathcal{N}_j} \mathcal{S}_{ig}(\boldsymbol{p}_j) = D_{cg}(\boldsymbol{p}_c) \quad \forall g \in \mathcal{M} \tag{9}$$

Or, equivalently, using the lifting notation:

$$\sum_i \hat{\mathcal{S}}_i(\boldsymbol{p}) = \hat{D}_c(\boldsymbol{p}) \tag{10}$$

Since the schedules are linear, we can write:

$$\sum_i \hat{\mathcal{S}}_i(\boldsymbol{p}) = \hat{\boldsymbol{A}}_c - \hat{B}_c\boldsymbol{p}_c$$
$$\left(\sum_j \hat{B}_j + \hat{B}_c\right)\boldsymbol{p} - \sum_j f_{j,L}\hat{B}_{j,f} = \hat{\boldsymbol{A}}_c - \hat{B}_c\boldsymbol{p}_c$$

Defining the matrices $M := \sum_j \hat{B}_j + \hat{B}_c$ and $M_f := \sum_j \hat{B}_{j,f}$, the market clearing system can be written as:

$$M\boldsymbol{p} = \boldsymbol{A}_c + M_f\boldsymbol{f}_L \tag{11}$$

Lemma 3.1 below shows that the system has a unique solution. We denote this unique solution as the *pricing function* mapping coefficient matrices to prices: $\boldsymbol{p} : B \to \boldsymbol{p}(B)$. This function is crucial: it embeds the information about competition and network interconnections.

Payoffs To complete the definition of the game, we have to define the payoffs. These are, in short, the profits, calculated in the prices that satisfy the market clearing conditions (9):

$$\pi_i(B) := \boldsymbol{p}_i'(B)\mathcal{S}_i(\boldsymbol{p}_i(B)) - \mathcal{S}_{\ell,i}(\boldsymbol{p}_i(B)) \tag{12}$$

$$= \boldsymbol{p}_i'(B)(B_i\boldsymbol{p}_i(B) - f_{iL}B_{i,f}) - \mathcal{S}_{\ell,i}(\boldsymbol{p}_i(B)) \tag{13}$$

So, formally, we give the following definition.

Definition 2.1.
A Supply and Demand Function Equilibrium (SDFE) is a Nash equilibrium of the game $G = (\mathcal{N}, (\mathcal{A}_i)_{i\in\mathcal{N}}, (\pi_i)_{i\in\mathcal{N}})$, where the players are the firms, actions are slopes, and the payoffs are the profits defined in (13)*.*

Example 1. Horizontal economy/Standard Supply Function Equilibrium

Consider the case of $N = 2$ firms, producing the same output good, without input-output connections (producing using only labor): $\boldsymbol{v}_i = 0$ for $i = 1, 2$. The demand function in this case is $D_c = A_c - B_c p_c$, where A_c, $B_c \in \mathbb{R}_+$. This is an instance of the Supply Function competition by Klemperer and Meyer (1989) (in the parametric case of the quadratic cost function).

Example 2. The vertical economy

The vertical economy illustrated in the Section 1 is a special case with $n = 2$ firms and $m = 2$ goods, where the technology satisfies: $\alpha_U = \alpha_D = 0$, $\boldsymbol{v}_U = 1, \boldsymbol{v}_D = (1, -1)$, and consumer demand satisfies: $B_c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Moreover, it is a limit case in which $k_i \to \infty$, so that the marginal costs are constant.

The next example is going to be very useful, because it is the most tractable case that allows to illustrate both the workings of the model and some implications in the next sections.

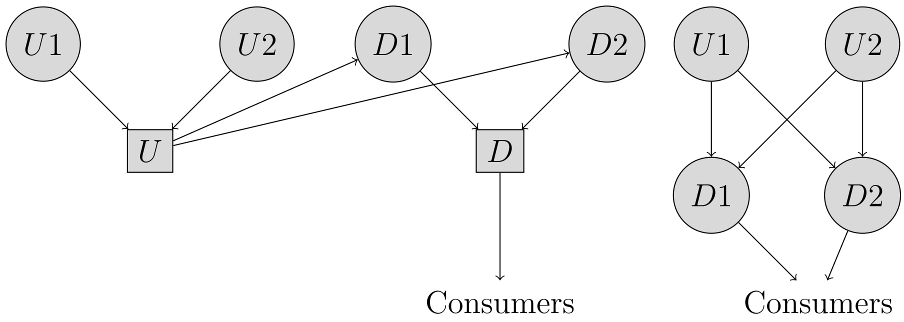


Figure 3: A layered supply chain. Left: bipartite representation, the squares represent goods, the circles firms. Right: firm-only representation.

Example 3 (**Supply chain with layers**)**.** A layered supply chain is a production structure in which firms are divided in N *layers*, and each layer produces one of the goods, as in Figure 3. There are n_i firms per layer. Layers are indexed from 1 to N moving upstream (we can consider the consumers as layer 0). Firms in layer $i+1$ sell to firms in layer i, firms in layer 0 sell their output to the consumer, firms in layer N only use labor for production. If $N=1$, we obtain the standard Supply Function equilibrium as in Klemperer and Meyer (1989), and the example above. For simplicity, assume that firms in each layer share the same parameters: so $f_{i,L}$, k_i and $f_{i,i+1}$ only depend on the layer i in which the firm is. So, $\boldsymbol{v}_i = (1, -f_{i,i+1})$ for each layer $i < N$, and $\boldsymbol{v}_N = 1$ for the last layer.

Each firm $i < N$ must submit a schedule $\mathcal{S}_i = (\mathcal{S}_i^{out}, -\mathcal{S}_i^{in})$, and the matrix $B_i \in \mathbb{R}^{2\times 2}$ satisfies:

$$B_i = \overline{B}_i \boldsymbol{v}_i \boldsymbol{v}_i' = \overline{B}_i \begin{pmatrix} 1 & -f_{i,i+1} \\ -f_{i,i+1} & f_{i,i+1}^2 \end{pmatrix}$$

or, equivalently: $\mathcal{S}_i^{out} = \overline{B}_i(p_i - f_{i,i+1}p_{i+1} - f_{i,L})$ and $\mathcal{S}_i^{in} = f_{i,i+1}\overline{B}_i(p_i - f_{i,i+1}p_{i+1} - f_{i,L})$

3 Solution and existence

First of all, the next Lemma makes sure that the pricing function and the payoffs defined in 2.2 is well-defined: the payoffs are indeed uniquely defined as a function of the slope coefficients B.

Lemma 3.1. If the production network is connected, the system (9) has a unique solution.

The proof is in the Appendix A.1.

3.1 Residual demand and price impact

The choice of a best reply of firm i to a profile B_{-i} can be written as:

$$\max_{B_i} \boldsymbol{p}_i(B)'(B_i\boldsymbol{p}_i(B) - B_{i,f}) - \ell_i(\boldsymbol{p}_i(B)) \tag{14}$$

subject to the technology constraints (8). However, the best reply problem can be more conveniently expressed in terms of the (inverse) residual demand subject to the market clearing conditions 9; and, analogously, in terms of the residual supply for the input market. We use the term *residual schedule* to indicate both the demand and supplies.

The key to understand the equilibrium conditions is to express the best reply problem using the residual schedule.

Lemma 3.2. 1. There exist a function $\boldsymbol{p}_i^r : \mathbb{R}^{d_i} \times \mathcal{A}_{-i} \to \mathbb{R}^{d_i}$, such that B_i^* solves the best reply problem (14) if and only if the quantity vector $\boldsymbol{q}_i^* = B_i^*\boldsymbol{p}_i^*(B_i^*, B_{-i}) - f_{i,L}\boldsymbol{v}_i$ solves:

$$\max_{\boldsymbol{q}_i, \tilde{\ell}_i} \boldsymbol{q}_i'\boldsymbol{p}_i^r(\boldsymbol{q}_i; B_{-i}) - \tilde{\ell}_i \tag{15}$$

subject to the technology constraints (8).

2. The function $\boldsymbol{p}_i^r$ is the *residual schedule*, and has the expression:

$$\boldsymbol{p}_i^r(\boldsymbol{q}_i; B_{-i}) = \Lambda_i(B_{-i})(\tilde{\boldsymbol{A}}_i(B_{-i}) - \boldsymbol{q}_i) \tag{16}$$

where $\Lambda_i(B_{-i}) : -\partial_{\boldsymbol{q}_i}\boldsymbol{p}_i^r = [(M - \hat{B}_i)^{-1}]_i$, and $\tilde{\boldsymbol{A}}_i$ is a vector that is also a function of B_{-i}.

Moreover, Λ_i is positive definite, and decreasing in all $\overline{B}_i$ in the positive semidefinite order.

The proof is in Appendix A.2.

The residual schedule, as in standard oligopoly models, represents the portion of demand and supply not met by competitors. Crucially, in an input-output setting, the residual schedule also contains information about the network position of firms. The coefficient matrix Λ_i is called the *price impact* (using a financial terminology) because it collects the slope coefficients of the (inverse) supply and demand schedules, describing what effect firm i has on its input and output prices. It is a measure of market power: the larger the price impact, the larger the surplus that the firm can extract from that buyer or seller.

The best reply problem expressed as in (15) is more convenient, because all the network interaction is summarized by the residual schedule. The optimization then, as in the example of Section 1, is analogous to the optimization of a monopolist, choosing the quantity (or the price) on the residual schedule to achieve its optimal profit. The key difference is that firms "implement" different prices by changing their schedule: and crucially this, in turn, changes the incentives of other firms to charge higher prices.

3.2 Existence and uniqueness

In this section I present the existence and uniqueness result for the Nash equilibrium.

Using the formulation of Lemma 3.2, and noting that the technology constraints (8) imply: $\boldsymbol{q}_i = q_i^{out}\boldsymbol{v}_i$, taking the first-order conditions we get:

$$\boldsymbol{v}_i'\boldsymbol{p}_i^r - q_i^{out}\boldsymbol{v}_i'\Lambda_i\boldsymbol{v}_i - \frac{1}{k_i}q_i^{out} = 0$$

Solving, we find that the optimal schedule is:

$$q_i^{out} = \mathcal{S}_i^{out}(\boldsymbol{p}_i) = \frac{\boldsymbol{v}_i'\boldsymbol{p}_i^r}{\boldsymbol{v}_i'\Lambda_i\boldsymbol{v} + 1/k_i}$$

So, the schedule is linear, and the coefficient satisfies:

$$\overline{B}_i = \left(\boldsymbol{v}_i'\Lambda_i\boldsymbol{v}_i + \frac{1}{k_i}\right)^{-1} \tag{17}$$

These are the equilibrium fixed-point equations. To understand the meaning, note that, for example, in the vertical economy of Section REFERENCE, we have $\boldsymbol{v}_i'\Lambda_i\boldsymbol{v}_i = \Lambda_i^{out} + \Lambda_i^{in}$, that is: the slope depends inversely on the sum of the price impacts on both the input and the output: we see a first effect of multilateral market power. In general, since $\boldsymbol{v}_i$ has negative elements for the inputs, the links between outputs and inputs are negatively weighted, so that:

$$\boldsymbol{v}_i'\Lambda_i\boldsymbol{v}_i = \Lambda_i^{out} + \boldsymbol{f}_i'\Lambda_i^{in}\boldsymbol{f}_i - 2\boldsymbol{f}_i'\Lambda_i^{out,in}$$

The next Theorem proves existence and uniqueness of the equilibrium.

Theorem 1.
There exist a unique Nash equilibrium of the game $\mathcal{G}$, and it is in pure strategies.[9] *Moreover, the equilibrium coefficients $\overline{B}_1, \dots, \overline{B}_n \in \mathbb{R}_+^n$ satisfy (17).*

The proof is in Appendix, A.3. The proof considers a modified game $\mathcal{G}'$ with action spaces $X_i = \mathbb{R}$ and payoffs: $U_i(x_1, \dots, x_n) = \ln \pi_i(e^{x_1}, \dots, e^{x_n})$. The new game corresponds to a reparameterization of the strategies of the original game, and a monotonic transformation of the payoffs. As such, any Nash equilibrium of the game $\mathcal{G}$ corresponds to one and only one Nash equilibrium of the game $\mathcal{G}'$. The game $\mathcal{G}' = (\mathcal{N}, (X_i, U_i)_{i\in\mathcal{N}})$ is a supermodular game, and thanks to the assumption of increasing marginal cost the strategy space is compact: $\overline{B}_i < k_i$, so the iteration of the best reply always converges.[10] Moreover, $\mathcal{G}'$ is also a potential game, and the potential is strictly concave: as a consequence, the game has a unique Nash equilibrium.

A first corollary is that in equilibrium we do not need to worry about exit of firms: profits are never negative.

[9] If Assumption 2 were assumption is violated, we get that the slopes of the firms competing tend to infinity, so technically the equilibrium does not exist; but from an economic perspective the limit is well defined, the involved firms simply behave as perfectly competitive. This is because, as highlighted by Klemperer and Meyer (1989), with constant marginal costs the supply function equilibrium behaves as price competition.

[10] This is not a necessary condition for the equilibrium. Indeed, the vertical economy example in the Section REFERENCE is an example where the equilibrium slopes are finite even when $k_i \to \infty$.

Corollary 3.1. In equilibrium profits are:

$$\begin{aligned}\pi_i(B) &= \overline{B}_i \boldsymbol{p}_i' \boldsymbol{v}_i \left(\boldsymbol{v}_i' \boldsymbol{p}_i - f_{iL}\right) - f_{iL} \overline{B}_i \left(\boldsymbol{v}_i' \boldsymbol{p}_i - f_{iL}\right) - \frac{1}{2k_i} \overline{B}_i^2 \left(\boldsymbol{v}_i' \boldsymbol{p}_i - f_{iL}\right)^2 \\ &= \overline{B}_i \left(1 - \frac{1}{2k_i} \overline{B}_i\right) \left(\boldsymbol{v}_i' \boldsymbol{p} - f_{iL}\right)^2 \end{aligned} \tag{18}$$

and, moreover, $\overline{B}_i < k_i$, so we get $\pi_i(\overline{B}) > 0$ for all firms i.

4 Equilibrium and the role of the network

So far, we identified the first-order conditions. Now we want to analyze what are the implications of the model in terms of firm's market power, particularly in relation to the network of input-output connections.

4.1 Markups and markdowns

The standard approach to measure market power is to look at the gap between price and marginal cost, or marginal revenue products.

Definition 4.1.
The total cost *of firm i is:* $C_i(q_i^{out}) = \sum_j p_j\left(\boldsymbol{q}_i\right) f_{ij} q_i^{out} + \frac{\alpha_i}{2}\left(q_i^{out}\right)^2$. *Define the (absolute)* markup *as* $\mu_i := p_i - \dfrac{\partial C_i}{\partial q_i^{out}}$.

The revenue product *of input g is:* $R_{ig}(q_{ig}) = p_i^{out} q_i^{out} - \sum_{j \neq g} p_j\left(\boldsymbol{q}_i\right) f_{ij} q_i^{out} - \frac{\alpha_i}{2}\left(q_i^{out}\right)^2$. *The* markdown on input g *is:* $\mu_{ig} := \dfrac{\partial R_{ig}}{\partial q_{ig}} - p_g$.

Lemma 4.1. The vector $\boldsymbol{\mu}_i = (\mu_i, -\mu_{ig})$ satisfies:

$$\boldsymbol{\mu}_i = q_i^{out} \Lambda_i \boldsymbol{v}_i, \tag{19}$$

where Λ_i is the price impact.

So, in the equilibrium of this model, each firm charges *both* a markup on the output and markdowns for each input. The magnitude of the markup and markdowns depends, not surprisingly, on the price impact: equation (19) is nothing beyond the standard Lerner equation connecting the slope (or elasticity) of demand to the price charged. The slope of the firm equilibrium schedule $\overline{B}_i$ depends inversely on the markups: the equilibrium equation (17) can be expressed also as:

$$\overline{B}_i = \frac{1}{(q_i^{out})^{-1}\boldsymbol{v}_i'\boldsymbol{\mu}_i + k_i^{-1}}$$

where $\boldsymbol{v}_i'\boldsymbol{\mu}_i = \mu_i + \sum_g f_{i,g}\mu_{i,g}$ aggregates the markup and markdowns: the larger this sum, the smaller the equilibrium slope. If the firm did take prices as given, the price impact would be zero and also the markups.

Markdowns are heterogeneous, and depend on the network position: this includes of course the number of competitors, but not only: also the number of indirectly connected customers or suppliers matters. The next subsection illustrates this through the simple examples of the vertical economy and the supply chain with layers. The subsection after connects the markups to the network structure in general.

4.2 Examples

4.2.1 Standard supply function equilibrium

The standard supply function equilibrium of Example 1 the price impact is simply the inverse slope of the residual demand:

$$\Lambda_i = \left(B_c + \sum_{j \neq i} B_j\right)^{-1} \tag{20}$$

4.2.2 The supply chain with layers

In the supply chain with layers the matrix M could be large if the number of layers N is large. So, to derive the price impact is more convenient to directly use the expression of the residual schedules. Let us focus on the case $B_c = f_{i,i+1} = 1$, for simplicity. For firms in layer 1, the slope of demand is $B_c + (n_1 - 1)B_1$. Firms in the upstream layer face a demand $n_1 B_1(p_1 - p_2) + (n_2 - 1)B_2(p_2 - p_3)$, where now is necessary to solve the first layer equations for p_1 as a function of p_2. Proceeding iteratively, we find the expression of the (direct) residual schedule:

$$q_i = \underbrace{(\overline{\Lambda}_i^{out})^{-1}(A - p_i)}_{\substack{\text{Demand}\\ \text{from customers}}} - \underbrace{(n_i - 1)B_i(p_i - p_{i+1})}_{\substack{\text{Supply of}\\ \text{competitors}}} \tag{21}$$

$$q_i = \underbrace{(\overline{\Lambda}_i^{in})^{-1} p_{i+1}}_{\substack{\text{Supply}\\ \text{from suppliers}}} - \underbrace{(n_i - 1)B_i(p_i - p_{i+1})}_{\substack{\text{Demand of}\\ \text{competitors}}} \tag{22}$$

where:

$$\overline{\Lambda}_i^{out} = \frac{1}{B_c} + \sum_{j<i} \frac{1}{n_j B_j} \tag{23}$$

$$\overline{\Lambda}_i^{in} = \sum_{j>i} \frac{1}{n_j B_j} \tag{24}$$

represent the "aggregate" price impact of firms in layer i, respectively, on the *output* price, and the *input* price.[11] These are intimately connected with the network: we can see that $\overline{\Lambda}_i^{out}$ is increasing with i, while $\overline{\Lambda}_i^{in}$ is decreasing.

Inverting the Jacobian of this, we find that the price impact matrix of firm i is equal to:

$$\Lambda_i = \begin{pmatrix} (\overline{\Lambda}_i^{out})^{-1} + (n_i - 1)B_i & -(n_i - 1)B_i \\ -(n_i - 1)B_i & (\overline{\Lambda}_i^{in})^{-1} + (n_i - 1)B_i \end{pmatrix}^{-1} \tag{25}$$

Define $Det = (\overline{\Lambda}_i^{in})^{-1}(\overline{\Lambda}_i^{out})^{-1} + ((\overline{\Lambda}_i^{in})^{-1} + (\overline{\Lambda}_i^{out})^{-1})(n_i - 1)B_i$ the determinant of Λ_i^{-1}. Finally, using the expression (19), we can compute the

[11] We might understand intuitively these equations, and in particular the term $\left((n_i - 1)B_i + (\overline{\Lambda}_i^{out} + \overline{\Lambda}_i^{in})^{-1}\right)^{-1}$ by noting that *horizontal* relationship (for example, the direct competitors in the same layer) imply a summation of the slope coefficients (the term $(n_i - 1)B_i$), while *vertical* relationships (across layers) imply an harmonic sum (the expressions for the price impacts in Equation (24)). This has an interesting analogy with the equations describing the electrical resistance: also in that case, when resistances are set in parallel (horizontally related), their total resistance is the sum of individual resistances, whereas when they are in sequence (vertically related), the total resistance is the harmonic sum.

markup vector:

$$
\begin{aligned}
\boldsymbol{\mu}_i &= q_i^{out}\Lambda_i \boldsymbol{v} \\
&= \frac{q_i^{out}}{Det}\begin{pmatrix} (\overline{\Lambda}_i^{in})^{-1} + (n_i-1)B_i & (n_i-1)B_i \\ (n_i-1)B_i & (\overline{\Lambda}_i^{out})^{-1} + (n_i-1)B_i \end{pmatrix}\boldsymbol{v} \\
&= \frac{q_i^{out}}{Det}\begin{pmatrix} (\overline{\Lambda}_i^{in})^{-1} \\ -(\overline{\Lambda}_i^{out})^{-1} \end{pmatrix} \\
&= q_i^{out}\begin{pmatrix} \dfrac{\overline{\Lambda}_i^{out}}{\overline{\Lambda}_i^{out} + (\overline{\Lambda}_i^{in} + \overline{\Lambda}_i)^{-1})(n_i-1)B_i} \\ -\dfrac{\overline{\Lambda}_i^{in}}{\overline{\Lambda}_i^{in} + (\overline{\Lambda}_i^{in} + \overline{\Lambda}_i)^{-1})(n_i-1)B_i} \end{pmatrix}
\end{aligned} \tag{26}
$$

It turns out that in the homogeneous case of $k_i = k$ and $n_i = n$, the "total" price impact $\overline{\Lambda}_i^{in} + \overline{\Lambda}_i^{out}$ is constant, and B_i too: as a consequence, only the ranking of $\overline{\Lambda}_i^{out}$ and $\overline{\Lambda}_i^{in}$ matters. The following Proposition makes this formal.

Proposition 1.
Suppose $k_i = k$ and $n_i = n$ for all layers i. In the Supply and Demand Function Equilibrium for the layered supply chain:

1. *the markups are larger the more upstream the layer is, while markdowns are larger the more downstream a layer is;*
2. *if $n_i \geq n_j$ firms in layer j have larger profits than firms in layer i.*

The intuition for the result above is simple: upstream firms perceive a smaller elasticity of the residual demand on output markets the more they are upstream, and so charge higher markups. The opposite happens with residual supply and markdowns. If n_i is constant across layers, the situation is completely symmetric, and so the increase in markups and decrease in markdowns exactly offset each other, and the firms all have the same profits. Hence, each layer extracts the same surplus. Instead, if some layer becomes more competitive (n_i is larger), the corresponding firms have lower profits.

4.2.3 The vertical economy

As illustrated in the introductory example in Section 1, the price impacts are:

$$\Lambda_U = \left(B_{c,U} + \left(\frac{1}{B_{c,D}} + \frac{1}{B_D} \right)^{-1} \right)^{-1} \tag{27}$$

$$\Lambda_D = \begin{pmatrix} 1/B_{c,D} & 0 \\ 0 & 1/(B_U + B_{c,U}) \end{pmatrix} \tag{28}$$

Hence, the first order conditions (17) reduce to (6) in Section 1.

This example shows that the price impact upstream Λ_U can be both larger or smaller than Λ_D, depending on the parameters. This is because in this economy, contrary to the supply chain with layers, the firm upstream also sells to consumers directly: having more customers increases the slope of demand, and this effect may counteract the pass-through effect.

4.3 The goods network

What can we say on the relation between the network position and market power in general? We summarize the discussion in the following remarks

Remark 4.1 (Prices as centralities)**.** From Equation REFERENCE, we can write:

$$\boldsymbol{p} = D^{-1}(I - G)^{-1}\boldsymbol{A}$$

where the diagonal matrix D has the slope of the excess supply $\sum_j B_{j,gg}$ as diagonal entry in position g, g, and the matrix G is defined as:

$$G_{i,gh} = \frac{-M_{g,h}}{\sum_j B_{j,gg}}$$

This can be thought of as the adjacency matrix of a weighted network, where nodes are *goods*, and a link is present when the price of h directly affects the quantities traded (to be precise, the excess supply) of g. The denominator is a normalization, measuring the effect of the price of g on the own excess supply. We label this network the *goods network*. Note that the weights of the links are endogenous and determined in equilibrium by the slopes of the schedules chosen. With this interpretation, the prices measure the Bonacich centrality in the goods network according to the weights vector $\boldsymbol{A}$. [12]

To understand better the interpretation of the link weights, let us consider the special case in which $\overline{B}_i = 1$ for all i, and the consumer demand satisfies $B_c = I$ (the identity).[13] In this case, the entries of the matrix M would be:

$$\begin{aligned}
M_{gg} &= |\text{firms trading } g, \text{except } i + 1| \\
M_{gh} &= -|\text{firms selling } h, \text{buying } g| - |\text{firms selling } g, \text{buying } h| \\
&\quad + |\text{firms buying both } g, h|,
\end{aligned}$$

so, the weight of the link between g and h is high when, among the firms trading h, many transform g and h or vice-verse, but not too many use both as inputs. This highlights that the network effect is strong when h, g have an input-output connection: in such a case, when the price of one goes up the other tends to increase too. Instead, an horizontal connection, since goods are perfect complements, because naturally in that case an increase in the quantity of one triggers a decrease in the price of the other: the weight can even be negative, if this effect is strong enough.

The network does not only offer an interpretation of the prices, but also of the price impacts and the slopes.

Remark 4.2 (Markups as centrality)**.** Define the *goods network ecluding firm i* as the goods network, where all the weights are computed as if $B_i = 0$, that is, excluding firm i. Call G_i the corresponding adjacency matrix. Then, the price impact can be written as:

$$\Lambda_i = [D_i^{-1}]_{\mathcal{N}(i)}[(I - G_i)^{-1}]_{\mathcal{N}(i)}$$

So that $\Lambda_{i,hg}$ is proportional to the number of *direct and indirect paths* between good g and h in the goods network excluding firm i.

[12]The goods network is undirected in terms of connections, but the weights may be asymmetric: this is an effect of the normalization.

[13]This in general is not the equilibrium, but it is always possible to find a configuration of k_i such that this is exactly the equilibrium.

As a consequence, the markup/markdown vector satisfies:

$$\boldsymbol{\mu}_i = q_i^{out}[D_i^{-1}]_{\mathcal{N}(i)}[(I - G_i)^{-1}]_{\mathcal{N}(i)}v_i = D_i^{-1}[(I - G_i)^{-1}\hat{\boldsymbol{v}}_i$$

So, the markup is proportional to the Bonacich centrality of good g in the goods network relative to firm i, with weights given by the vector $\hat{\boldsymbol{v}}_i$. The markup also depends on the cross effect of the output price on the input. However, since $\hat{\boldsymbol{v}}_i$ has negative elements, this effect is weighted negatively: this is because when the cross-effect is large, a large markup (low quantity), triggers an increase of the input price, that decreases the markup.

Finally, since the slopes depend inversely on $\hat{\boldsymbol{v}}_i'\boldsymbol{\mu}_i$, we conclude that the equilibrium slopes depend inversely on the weighted sum of the centralities.

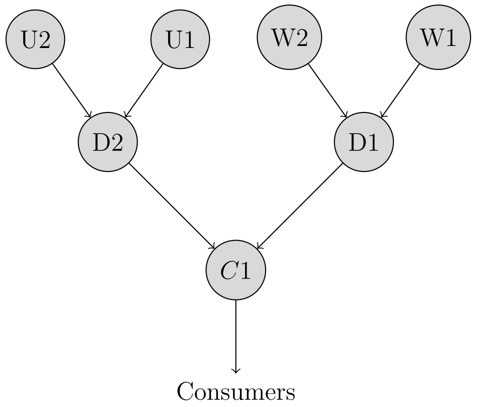
Definition 4.2.
Define the goods network relative to firm i *as the network* $(\mathcal{M}, \mathcal{L})$ *where:*

1. *the nodes are the goods,* $\mathcal{M}$*;*
2. *two goods-nodes* g, h *are linked if there is at least a firm trading both,* ***apart from*** i*:* $(h, g) \in \mathcal{L}$ *if and only if there is* $j \in \mathcal{N}$ *such that* $h \in \mathcal{N}_i$ *and* $g \in \mathcal{N}_i$*;*
3. *The adjacency matrix of the network relative to firm* i *is the matrix* G_i *that has weights:*

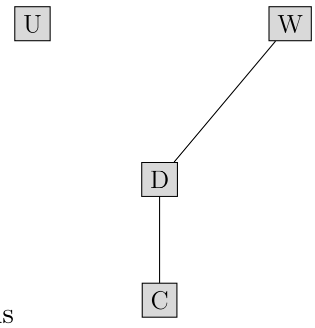
$$G_{i,gh} = \frac{-M_{-i,gh}}{\sqrt{D_{i,gg}D_{i,hh}}}$$

Example 4 ("Tree" network)**.** Consider the production network depicted in Figure 4a. There are 4 goods: U, W, D and C. Each good except C is produced by two firms: e.g. U is produced by $U1$ and $U2$. In Figure 4b is represented graphically the *goods network* of this economy relative to firm $D2$: the network is disconnected, because without firm $D2$ there is no firm trading both goods U and D.

Since the goods network is disconnected, the price impact matrix is a block-diagonal matrix. In this case, since firms have only one input, it is



(a) A production network where 4 goods are traded: U, D, W and C. Each good except good C is sold by 2 firms.



(b) The goods network relative to firm $D2$.

actually diagonal, and is:

$$\Lambda_i = \begin{pmatrix} \Lambda_i^{out} & 0 \\ 0 & \Lambda_i^{in} \end{pmatrix}$$

Point 1 of Theorem REFERENCE then means that Λ_i^{out} is equal to the slope of the schedule of firm C minus the supply of $D1$, times the network effect, which is the number of cycles of the output link in the reduced graph; and, similarly, for Λ_i^{in}. The number of cycles is a measure of size and, the higher the weights REFERENCE, the higher the measure. In this example, the cycles centred in good U are just 1 (the trivial cycle), so the indirect effect is equal to 1; while in the output good is a higher number. Point 2 then allows to conclude that in this case. since Λ_i is diagonal, the markup is simply proportional to the price impact.

For a more general case, let us consider the supply chain with layers, in which the price impact is given in Equation REFERENCE. The main difference is that now the network is connected, and so a change in the output quantity will also affect the input price. Since M is an M-matrix, it is easy to conclude here that Λ_i has all positive entries. So, an increase in output quantity will decrease both the output price, directly, but also the input price, indirectly. Why? An increase in the output price will make the competitor sell more, so buy more, and so trigger an increase in the input price. This is why $\Lambda_{i,gh} > 0$. By the same reason, the price impact on the output is still measured by the cycles centered in good 1, but now the cycles involve also the inputs. Again, this is because the network is connected. So, the markup is proportional to the cycles centered in 1, minus the measure of direct and indirect links between 1 and 2.

5 The role of multilateral market power

5.1 General price impacts

The key feature of the model studied so far is that firms have multilateral market power: they can affect prices in all the markets they are involved in. What are the implications of multilateral market power? To answer this question, in this section I introduce a model that simultaneously generalizes the supply and demand function competition and various other classic models of oligopolistic competition, with and without networks. This allows us to do comparative statics on *market power*, comparing the model of the previous sections with an analogous model in which firms are price-takers on input markets.

Definition 5.1.
Consider a profile of functions $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$*, where:*

$$\Lambda_i : B_{-i} \to \Lambda_i(B_{-i}) \in \mathbb{R}^{d_i \times d_i}$$

such that for all i*:*

1. Λ_i *is continuous;*
2. Λ_i *is positive semidefinite;*
3. Λ_i *is decreasing in the profile* B *in the positive semidefinite ordering.*

A Generalized SDFE is a profile of schedules $B^* = (B_1^*, \ldots, B_n^*)$ *such that each* B_i^* *solves the best reply problem* (15)*, but where the residual schedule satisfies:*

$$\partial_{\boldsymbol{q}_i} \boldsymbol{p}_i^r(\boldsymbol{q}_i, B_{-i}) = \Lambda_i(B_{-i}) \quad \forall B_{-i} \, \forall i \in \mathcal{N}$$

The Supply and Demand function competition is a Generalized SDFE, because Λ_i derived in Lemma 3.2 is continuous and decreasing. The interest of the Generalized SDFE is that many other standard models are also special cases. For example, Walrasian equilibrium is the special case where $\Lambda_i = 0$ for each i. Also Cournot oligopoly is a special case: consider the setting of the standard Supply Function Equilibrium of Example 1 and change $\Lambda_i = \dfrac{1}{B_c}$ in Equation (20). What this means is that firms behave as if competitors have choosen schedules with constant slope (set B_j to 0 in the equation): but schedules with constant slopes are fixed quantities, as in Cournot. Indeed, it can be checked that this Generalized SDFE yields exactly the same equilibrium quantities and price as the Cournot oligopoly with the same parameters of Example 1. The case of Bertrand is analogous, provided we use differentiated products.[14] What is perhaps more striking is that also some *sequential* models are also special cases of the Generalizes SDFE, as we argue below.

The next Theorem proves existence of an equilibrium, and the fundamental comparative statics result on the price impacts.

Theorem 2. *1. A generalized SFE exists. Moreover, it is a game of strategic complements, and as such it always has a maximal and a minimal equilibrium (possibly identical).*

2. Consider two models in which the profile of price impact functions are, respectively Λ^1 and Λ^2, such that for each profile B we have $\Lambda^1_i(B_{-i}) \geq \Lambda^2_i(B_{-i})$ for all firms i (in the p.s.d. ordering). Then, in the maximal and the minimal equilibria, the slope coefficients are lower the first model: $(B^1)^*_i \leq (B^2)^*_i$ (in the p.s.d. ordering) for each firm i.

The key intuition both of part 1) and 2) comes once again by strategic complementarity: a lower price impact means higher slopes, that in turn trigger higher best response, and equilibrium slopes.

The proof is in Appendix C.1.

The previous Theorem looks at comparative statics with respect to

[14] This is because in my model each good has a unique price, and so the standard homogeneous goods Bertrand competition violates this assumption.

price impacts. If there is only one final price, it is possible to extend the comparative static exercise looking at the effect on the final price.

Corollary 5.1. Consider two profiles of price impact functions Λ^1 and Λ^2 ordered as in Theorem 2, part 2. In any network such that the consumer only consumes one good $\mathcal{C} = \{c\}$, the price of the final good is higher in the model with smaller price impacts Λ^2.

This is the tool with which we can explore the effect of different assumptions on multilateral market power.

5.2 Comparison with unilateral and local market power

As discussed in the Literature section, many papers in the production network literature assume as a simplification that input prices are taken as given, and that prices in other markets are taken as given. In the Generalized SDFE model, it is easy to embed these two assumptions, with assumptions on the functional form of the price impact. Let us first define these two assumptions precisely.

Definition 5.2. 1. *The model with* unilateral market power *is a Generalized SDFE in which firms take input prices as given.*

2. *The model with* local market power *is a Generalized SDFE where firms take as given the prices in the markets in which they are not directly involved in.*

The next Theorem is the main result of the Section.

Theorem 3.
The models of Definition 5.2 are special cases of the Generalized SFE, for different choices of the price impact functions Λ_i:

1. ***Unilateral market power**: for all firms i using intermediate inputs:*[15]

$$\Lambda_i^{unilateral} = \begin{pmatrix} (M - \hat{B}_i)_{ii}^{-1} & \mathbf{0}' \\ \mathbf{0} & 0 \end{pmatrix}$$

[15]It turns out that this approach is exactly the one that allows to recover the Sequential Oligopoly as a special case of the model.

2. ***Local market power:***

$$\Lambda_i^{local}(B_{-i}) = (M_i - B_i)^{-1}$$

Moreover, in both cases, the price impacts are smaller, that is $\forall i \in \mathcal{N}\, \forall B_{-i}$ *we have* $\Lambda_i^{local}(B_{-i}) \leq \Lambda_i^{multilateral}(B_{-i})$ *and* $\Lambda_i^{unilateral}(B_{-i}) \leq \Lambda_i^{multilateral}(B_{-i})$

Now, to explore the effect of multilateral market power, is sufficient to consider the price impact functions of the previous Theorem, and compare them with the baseline. For example, consider the setting of the vertical economy of Section 1; and suppose we want to compute the Generalized SDFE with unilateral market power. According to Theorem 3, we simply have to modify the equilibrium equations by changing the price impact of Equation (28) to:

$$\Lambda_D^{sequential} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

while Λ_U, having no (non-labor) intermediate inputs, is unaffected. It is clear that $\Lambda^{\text{unilateral}} \leq \Lambda^{\text{multilateral}}$; and thanks to Theorem 2, we know that in equilibrium this implies that the slopes are higher in the unilateral model.

Using Corollary 5.1, we can also conclude:

Corollary 5.2. For any network in which such that there is only one final good, with unilateral or local market power the final price is smaller than in the benchmark SDFE with multilateral market power.

This is the main result on the role of multilateral market power. We express it restricting market power on inputs because it is the standard assumption, but if we were to restrict market power on outputs and set $\Lambda_i^{out} = 0$, the exact same conclusion would hold.

Remark 5.1 (**Sequential Monopoly is a Generalized SDFE**)**.** The most standard way to model price setting in the context of the vertical economy is perhaps the Sequential Monopoly à la Spengler (1950), that is a sequential game where firms set output prices sequentially, starting upstream with U and then D, and D (by construction) takes the input price p_D as given. It turns out that the first-order conditions of the Sequential

Monopoly thus defined imply *exactly* the same equilibrium price and quantity of the Generalized SDFE with unilateral market power. To understand why, let's write the first-order conditions of the sequential model. Since it is a monopoly, setting quantities or prices is exactly equivalent. By backward induction start from firm D. The inverse demand is as in Section 1: $p_D = A - q_D$. Maximizing the profit of D while taking p_U as given produces the FOCs for the downstream firm:

$$p_D - p_U + \frac{\partial p_D}{\partial q_D} q_D = 0 \tag{29}$$

Notice that this is precisely the same as Equation (5b) when $\frac{\partial p_U}{\partial q_D} = 0$, that is the FOC of the Generalized SDFE with unilateral market power. The mechanism is the same: since the firm does not internalize the price impact on the input, that term disappears from the FOC.

To understand why also the FOC for the upstream firm mimic the SDFE is a bit more subtle. In the sequential monopoly model, the (inverse) demand for firm U, in equilibrium, is given by the equilibrium choice of firm D as a function of p_U. But this means exactly to use equation (29) and the consumer demands, to back up p_U^r: this is exactly the same as solving the market clearing conditions for a given choice of schedule of firm D. So, we get that the equilibrium demand for firm U is exactly the same in the Sequential Competition, and in the Generalized SDFE with unilateral market power!

In the Supplementary Appendix, I prove that the analogy does not stop at the Sequential Monopoly, but it extends also to Sequential Cournot, that is a Generalized SDFE with unilateral market power, with the additional "Cournot" assumption that firms consider flat the schedules chosen by direct competitors.

We conclude the section showing that multilateral market power also affects the way the surplus is split, in addition to the total size. In the case of the layered supply chain, we can make precise characterizations.

Proposition 2. 1. *If firms take the* input *price as given, markups are still increasing going* upstream*, while there are no markdowns: as a consequence, profits are increasing upstream.*

2. *If firms take the* output *price as given, then markdowns are increasing going* downstream*, while there are no markups: as a consequence, profits are increasing downstream.*

If firms instead do not internalize their effect on input prices, but only outputs, the symmetry is broken, because firms consider the effect of network position on the elasticitiy of demand only on, e.g., the output side.

These results yield important insights on the hidden consequences of using models in which competition is artificially constrained to be unilateral. If such a modeling strategy is not motivated by the specifics of the market studied, but is just an assumption imposed for tractability, as in production network models, the result above suggests that implication for both the total amount of distortions due to market power and the relative ranking of market power among firms can be severely changed. The supply and demand function equilibrium provides a setting in which the modeler does not have to choose on which side of the market firms can affect prices, rather the price impact is an additional prediction that can be asked to the model.

Example 5. Vertical mergers can be welfare improving or not

For a particularly stark example, consider an instance of the Supply chain with layers, with 2 layers, with 1 firm in the upstream sector 2 and n_1 firms in the downstram sector, 1. Suppose after a merger between the firm in 1 and a firm in 0 the merged firm does not sell its intermediate good to others but it keeps it all to produce the final output. Then all other firms in 1 cannot produce anymore, and we are left with a monopoly, as shown in Figure 5. The monopoly price in the after-merger setting is:

$$p^M = A\left(B_c + \frac{1}{1 + 1/B_c}\right)^{-1}$$

where $B^M = \frac{B_c}{1+B_c}$ is the equilibrium coefficient of the supply of the only firm.

In the pre-merger equilibrium instead the final price is:

$$p = \frac{A}{B_c + \frac{n_1 B_1 B_2}{n_1 B_1 + B_2}} = A\left(B_c + \frac{1}{1 + \frac{2}{n_1 B_1}}\right)^{-1}$$

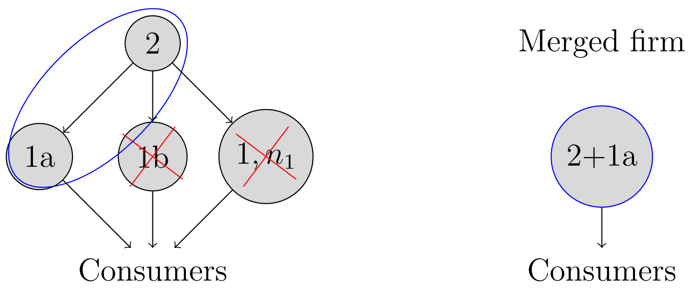


Figure 5: Left: pre-merger economy. The blue circle indicated the merging firms 2 and $1a$. Right: the economy after the merger: $1b$ and $1c$ are driven out of the market because the merged firm does not sell them the necessary input anymore, and the merged firm becomes a monopolist.

where B_1 and B_2 are as usual the coefficients of the equilibrium supply and demand functions, and the last equality is obtained using the best reply equation for B_2. Hence we get that the price is higher after the merger if and only if $2B_c < n_1 B_1$. The expression shows the trade-off between double marginalization, represented by the factor of 2 that appears because the pre-merger economy is a line with 2 steps, and the extent of foreclosure, represented by $n_1 B_1$, that measures how much competition is lost after the merger:

$$\underbrace{2}_{\text{decreased double marginalization}} \times B_c < \underbrace{n_1 B_1}_{\text{extent of foreclosure}}$$

If $B_c > 1$, since $B_1 < 1$, for $n_1 = 2$ the merger is welfare-improving. Since the RHS goes to infinity for n_1 sufficiently large, the merger is welfare reducing. In particular, we can identify a n_* such that the merger is welfare-decreasing if $n > n_*$ (because the foreclosure effect is stronger), and welfare improving if $n < n_*$. Such value is defined implicitly by $B_c = n_* B_0^{multilateral}(n_*)$.

Now consider the model with unilateral market power. We can define a similar threshold n^*, defined by $B_c = n^* B_1^{local}(n^*)$. By Theorem 2 for any n_1, $B_1^{\text{unilateral}}(n_1) > B_1^{\text{multilateral}}(n_1)$, so that $n_* < n^*$. Hence, it follows that for $n \in (n_*, n^*)$ the merger is welfare decreasing with multilateral market power, but welfare-increasing with unilateral market power.

6 Conclusion

This paper provides a way to model oligopoly in general equilibrium as a game in which firms fully internalize their position in the supply chain and have market power both over inputs and outputs, in an endogenously determined way. I show that such features are desirable in a input-output model with market power: if absent, both the aggregate and the relative ranking of distortions due to imperfect competitions is crucially affected. This suggests that, when modeling complex networks of large firms with market power, simplifying assumptions might affect in a sizable way the results.

A further interest of the competition in schedules framework is that it is a standard model for procurement auctions (Holmberg et al., 2019; Klemperer and Meyer, 1989; Ausubel et al., 2014), where the consumer is the auctioneer. The results developed can help shed light on price formation in procurement auctions where the bidders are simply the last stage of a potentially complex supply chain. The exploration of the implications of this for design are an interesting avenue for further research.

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Appendix

A Proofs of Section 3

A.1 Proof of Lemma 3.1

We prove that it is always possible to invert the market clearing conditions (11).

Consider the quadratic form $\boldsymbol{x}'M\boldsymbol{x}$. This is equal to $\boldsymbol{x}'M\boldsymbol{x} = \sum_i \boldsymbol{x}'\hat{B}_i\boldsymbol{x} + \boldsymbol{x}'\hat{B}_c\boldsymbol{x} = \sum_i \boldsymbol{x}_i'B_i\boldsymbol{x}_i + \boldsymbol{x}_c'B_c\boldsymbol{x}_c$, where, as for the prices, we denote $\boldsymbol{x}_i = (x_g)_{g\in\mathcal{N}_i}$.

Restrict attention to a subset of n firms, chosen such that each firm produces a distinct good: for each good g, denote i_g the firm producing g that is chosen. Define F the matrix with elements $f_{i_g h}$. By the assumption of viability, $I - F'$ is an M-matrix, and in particular is invertible (Horn et al., 1994). Moreover, the columns of $I - F'$ are the $\hat{\boldsymbol{v}}_{i_g}$ vectors: so, they must be linearly independent. So, there are at least n linearly independent $\hat{\boldsymbol{v}}_i$. So, there are at least n linearly independent $\hat{\boldsymbol{v}}_i$ vectors. Now: $\boldsymbol{x}'M\boldsymbol{x} = \boldsymbol{x}_c'B_c\boldsymbol{x}_c + \sum_i \overline{B}_i\boldsymbol{x}'\hat{\boldsymbol{v}}_i\hat{\boldsymbol{v}}_i'\boldsymbol{x}$. For this to be zero, it means that $\boldsymbol{x}$ must either be zero, or be orthogonal to $\hat{\boldsymbol{v}}_i$ for every i. But since they include a basis, if follows that $\boldsymbol{x} = 0$. If $\boldsymbol{x}'\left(M - \hat{B}_i\right)\boldsymbol{x} = 0$ by analogous reasoning $\boldsymbol{x}$ must be orthogonal to all $\hat{\boldsymbol{v}}_j$ for $j \neq i$. So, either it is 0, or is parallel to $\hat{\boldsymbol{v}}_i$. Then, with an induction argument analogous to the one below, we obtain the thesis.

□

A.2 Proof of Lemma 3.2

Consider the system 11. It can equivalently be rewritten as:

$$\hat{\boldsymbol{q}}_i + (M - \hat{B}_i)\boldsymbol{p} = \boldsymbol{A}_c \tag{30}$$

We know from Lemma 3.1 that $M - \hat{B}_i$ is positive definite, so we can invert it and write:

$$\boldsymbol{p}^r(\boldsymbol{q}_i) = (M - \hat{B}_i)^{-1}(\boldsymbol{A} - \hat{\boldsymbol{q}}_i)$$

Reordering the equations so to have all the rows associated with inputs and ouputs of i first, we can write the matrix M in blocks as:

$$M - \hat{B}_i = \begin{pmatrix} M_i - B_i & M_{i,-i} \\ M'_{i,-i} & M_{-i} \end{pmatrix}$$

where M_i is the submatrix relative to input and output goods of i, M_{-i} the submatrix relative to goods that are neither inputs nor outputs of i, and $M_{i,-i}$ is the off-diagonal block. Using the rule for block matrix inversion:

$$\boldsymbol{p}_i^r = [(M - \hat{B}_i)^{-1}]_i \left(\boldsymbol{A}_i - \boldsymbol{q}_i - M_{i,-i}(M_{-i})^{-1}\boldsymbol{A}_{-i}\right)$$

Defining $\Lambda_i := [(M - \hat{B}_i)^{-1}]_i$ and $\tilde{\boldsymbol{A}}_i := \boldsymbol{A}_i - M_{i,-i}(M_{-i})^{-1}\boldsymbol{A}_{-i}$ we obtain the thesis.

□

A.3 Proof of Theorem 1

To derive the game $\mathcal{G}'$ as defined in the text, first let us check that we can limit ourselves to schedules of the form REFERENCE. Solving the maximization problem in Lemma 3.2, we get:

$$\mathcal{S}_i^{out}(\boldsymbol{p}_i) = \left(\boldsymbol{v}_i'\Lambda_i\boldsymbol{v}_i + 1/k_i\right)^{-1}\left(\boldsymbol{v}_i'\boldsymbol{p}_i - f_{iL}\right)$$

As a consequence, we have B_i must have the form $\overline{B}_i\boldsymbol{v}_i\boldsymbol{v}_i'$ for some $\overline{B}_i \in \mathbb{R}_+$. So, the payoff of firm i can be rewritten as:

$$\tilde{\pi}_i(\overline{B}_i, \overline{B}_{-i}) = \overline{B}_i\left(\boldsymbol{v}_i'\boldsymbol{p}_i - f_{iL}\right)$$

Now, we consider the game $\mathcal{G}'$ with the payoffs defined in the main text. Since both $\ln(\cdot)$ and $exp(\cdot)$ are monotone, a profile $\overline{B}$ is a pure Nash equilibrium of $\mathcal{G}$ if and only if the profile $x = (\ln\overline{B}_1, \ldots, \ln\overline{B}_n)$ is a pure Nash equilibrium of $\mathcal{G}'$. It follows that the Nash equilibrium of $\mathcal{G}'$ is unique if and only if the Nash equilibrium of $\mathcal{G}$ is unique.

Existence The best reply equation (17) shows that $\overline{B}_i \in [0, k_i]$, and is continuous. So, by Brouwer's fixed point theorem, there exist an equilibrium.

Potential To show that the modified game $\mathcal{G}'$ is a potential game, we show that the second cross-derivatives of the payoffs are equal. Since the supply is $\mathcal{S}_i = \overline{B}_i \boldsymbol{v}_i \boldsymbol{v}_i' \boldsymbol{p}_i$, the profit can be rewritten as:

$$\begin{aligned}
\pi_i &= \boldsymbol{p}_i' \mathcal{S}_i - \frac{\alpha_i}{2} \left(\mathcal{S}_i^{out}\right)^2 \\
&= \overline{B}_i \boldsymbol{p}_i' \boldsymbol{v}_i \boldsymbol{v}_i' \boldsymbol{p}_i - \frac{\alpha_i}{2} \overline{B}_i^2 \left(\boldsymbol{v}_i' \boldsymbol{p}_i\right)^2 \\
&= \overline{B}_i \left(1 - \frac{\alpha_i}{2} \overline{B}_i\right) \left(\boldsymbol{v}_i' \boldsymbol{p}_i\right)^2 \\
&= \overline{B}_i \left(1 - \frac{\alpha_i}{2} \overline{B}_i\right) \left(\hat{\boldsymbol{v}}_i' \boldsymbol{p}\right)^2
\end{aligned}$$

To compute the derivative of the payoffs, we must differentiate the matrix M^{-1}. Its derivative is:

$$\frac{\partial}{\partial \overline{B}_i} M^{-1} = -M^{-1} \mathrm{d} \left(\sum_j \overline{B}_j \hat{\boldsymbol{v}}_j \hat{\boldsymbol{v}}'_j + \hat{B}_c \right) M^{-1} = -M^{-1} \hat{\boldsymbol{v}}_i \hat{\boldsymbol{v}}'_i M^{-1}$$

Moreover, $\hat{\boldsymbol{v}}_i' \boldsymbol{p} = \hat{\boldsymbol{v}}_i' M^{-1} \boldsymbol{A}$. So:

$$\frac{\partial}{\partial \overline{B}_i} \hat{\boldsymbol{v}}_j' \boldsymbol{p} = -\hat{\boldsymbol{v}}_j M^{-1} \hat{\boldsymbol{v}}_i \hat{\boldsymbol{v}}_i' M^{-1} \boldsymbol{A} = -\hat{\boldsymbol{v}}_j M^{-1} \hat{\boldsymbol{v}}_i \hat{\boldsymbol{v}}_i' \boldsymbol{p}_i \tag{31}$$

Using this, we find that the derivative of the profit is:

$$\begin{aligned}
\frac{\partial \pi_i}{\partial \overline{B}_i} &= \left(1 - \alpha_i \overline{B}_i\right) \left(\hat{\boldsymbol{v}}_i \boldsymbol{p}\right)^2 - 2\overline{B}_i \left(1 - \frac{\alpha_i}{2} \overline{B}_i\right) \left(\hat{\boldsymbol{v}}_i \boldsymbol{p}\right) \hat{\boldsymbol{v}}_i M^{-1} \hat{\boldsymbol{v}}_i \hat{\boldsymbol{v}}_i' \boldsymbol{p}_i \\
&= \left(1 - \alpha_i \overline{B}_i\right) \left(\hat{\boldsymbol{v}}_i' \boldsymbol{p}\right)^2 - 2\overline{B}_i \left(1 - \frac{1}{2} \alpha_i \overline{B}_i\right) \left(\hat{\boldsymbol{v}}_i' \boldsymbol{p}\right)^2 \hat{\boldsymbol{v}}_i' M^{-1} \hat{\boldsymbol{v}}_i \\
&= \overline{B}_i \left(1 - \frac{1}{2} \alpha_i \overline{B}_i\right) \left(\hat{\boldsymbol{v}}_i' \boldsymbol{p}\right)^2 \left(\frac{1 - \alpha_i \overline{B}_i}{\overline{B}_i \left(1 - \frac{1}{2} \alpha_i \overline{B}_i\right)} - 2\hat{\boldsymbol{v}}_i' M^{-1} \hat{\boldsymbol{v}}_i \right)
\end{aligned}$$

Notice that with our reparameterization $\overline{B}_i = e^{x_i}$ and $\frac{\partial U_i}{\partial x_i} = \frac{\partial \ln \pi_i}{\partial \ln \overline{B}_i}$. Since

$\pi_i = \overline{B}\left(1 - \frac{\alpha_i}{2}\overline{B}_i\right)(\hat{\boldsymbol{v}}_i \boldsymbol{p})^2$, the derivative of U_i becomes:

$$\begin{aligned}\frac{\partial U_i}{\partial x_i} &= 1 - \frac{1}{2}\frac{\alpha_i e^{x_i}}{\left(1 - \frac{1}{2}\alpha_i e^{x_i}\right)} - 2e^{x_i}\hat{\boldsymbol{v}}_i' M^{-1}\hat{\boldsymbol{v}}_i \\ &= 1 - \frac{1}{2}\frac{\alpha_i \overline{B}_i}{\left(1 - \frac{1}{2}\alpha_i \overline{B}_i\right)} - 2\overline{B}_i\hat{\boldsymbol{v}}_i' M^{-1}\hat{\boldsymbol{v}}_i\end{aligned}$$

Using again 31:

$$\frac{\partial^2 U_i}{\partial x_j \partial x_i} = \begin{cases} 2\overline{B}_i\overline{B}_j\hat{\boldsymbol{v}}_i' M^{-1}\hat{\boldsymbol{v}}_j\hat{\boldsymbol{v}}_j' M^{-1}\hat{\boldsymbol{v}}_i = 2\left(\hat{\boldsymbol{v}}_i' M^{-1}\hat{\boldsymbol{v}}_j\right)^2 & i \neq j \\ = -\frac{\alpha_i}{2}\frac{1}{\left(1 - \frac{1}{2}\alpha_i\overline{B}_i\right)^2} - 2\hat{\boldsymbol{v}}_i' M^{-1}\hat{\boldsymbol{v}}_i + 2\left(\hat{\boldsymbol{v}}_i' M^{-1}\hat{\boldsymbol{v}}_i\right)^2 & i = j \end{cases}$$

Since $\frac{\partial^2 U_i}{\partial x_j \partial x_i} = \frac{\partial^2 U_j}{\partial x_i \partial x_j}$, the game is a potential game.

This means that there exists a function Φ such that:

$$\frac{\partial \Phi}{\partial x_i} = \frac{\partial U_i}{\partial x_i}$$

For each i. In particular, this means that $\frac{\partial^2 \Phi}{\partial x_i \partial x_j} = \frac{\partial^2 U_i}{\partial x_i \partial x_j}$. So, even without knowing the expression of Φ, we can compute its Hessian matrix: denote it as H.

Uniqueness Now, we prove that the potential is strictly concave. This proves that the game can have at most one Nash equilibrium. To prove it, we prove that the Hessian matrix H is negative definite, by proving that

$-H$ is strictly diagonally dominant. Sum the off-diagonal entries:

$$
\begin{aligned}
\sum_{i\neq j} |H_{ij}| &= \sum_{i\neq j} H_{ij} \\
&= \sum_{i\neq j} 2\overline{B}_j\overline{B}_i \left(\hat{\boldsymbol{v}}_i' M^{-1}\hat{\boldsymbol{v}}_j\right)^2 \\
&= 2\overline{B}_j\hat{\boldsymbol{v}}_j' M^{-1}\left(\sum_{i\neq j}\overline{B}_i\hat{\boldsymbol{v}}_i\hat{\boldsymbol{v}}_i'\right) M^{-1}\hat{\boldsymbol{v}}_j \\
&= 2\overline{B}_j\hat{\boldsymbol{v}}_j' M^{-1}\left(M-\hat{B}_c-\overline{B}_j\hat{\boldsymbol{v}}_j\hat{\boldsymbol{v}}_j'\right) M^{-1}\hat{\boldsymbol{v}}_j \\
&= 2\overline{B}_j\hat{\boldsymbol{v}}_j' M^{-1}MM^{-1}\hat{\boldsymbol{v}}_j - 2\overline{B}_j\hat{\boldsymbol{v}}_j' M^{-1}\left(\hat{B}_c+\overline{B}_j\hat{\boldsymbol{v}}_j\hat{\boldsymbol{v}}_j'\right) M^{-1}\hat{\boldsymbol{v}}_j \\
&< 2\overline{B}_j\hat{\boldsymbol{v}}_j' M^{-1}\hat{\boldsymbol{v}}_j - 2\overline{B}_j\hat{\boldsymbol{v}}_j' M^{-1}\left(\overline{B}_j\hat{\boldsymbol{v}}_j\hat{\boldsymbol{v}}_j'\right) M^{-1}\hat{\boldsymbol{v}}_j \\
&= 2\overline{B}_j\hat{\boldsymbol{v}}_j' M^{-1}\hat{\boldsymbol{v}}_j\left(1-\overline{B}_j\hat{\boldsymbol{v}}_j' M^{-1}\hat{\boldsymbol{v}}_j\right) \\
&= -H_{jj}
\end{aligned}
$$

where the strict inequality is because $\hat{B}_c$ is positive semidefinite, and there must be at least a path from each firm j to the consumer, so that $\left[M^{-1}\hat{\boldsymbol{v}}_j\right]_c \neq 0$. Since the expression above is a sum of positive terms, it follows that $H_{jj} < 0$, and $-H_{jj} > \sum_{i\neq j}|H_{ij}|$, so $-H$ is strictly diagonally dominant, so is negative definite. Hence, Φ is concave and the Nash equilibrium is unique.

Moreover, since the diagonal of H is negative, it follows that the payoffs are concave, so the FOCs (17) are necessary and sufficient for the equilibrium. Moreover, the game is a supermodular game, so the unique Nash equilibrium is also the unique rationalizable action profile. □

B Proofs of Section 4

B.1 Proof of Lemma 4.1

The revenue product generated by input g is:

$$\begin{aligned}
R_k(q_{ig}) &= \boldsymbol{q}_i'\boldsymbol{p}_i - \frac{\alpha_i}{2}\left(q_i^{out}\right)^2 + p_g q_{ig} \\
&= q_i^{out}\boldsymbol{v}_i'\boldsymbol{p}_i - \frac{\alpha_i}{2}\left(q_i^{out}\right)^2 + p_g q_{ig} \\
&= \frac{q_{ig}}{f_{ig}}\boldsymbol{v}_i'\boldsymbol{p}_i\left(\frac{q_{ig}}{f_{ig}}\boldsymbol{v}_i\right) - \frac{\alpha_i}{2}\left(\frac{q_{ig}}{f_{ig}}\right)^2 + p_g q_{ig}
\end{aligned}$$

We first compute the marginal cost and the marginal revenue product:

$$\frac{\partial C}{\partial q_i^{out}} = \sum_j p_j\left(q_i^{out}\boldsymbol{v}_i\right)f_{ij} - \sum_j\left[\Lambda_i\boldsymbol{v}_i\right]_j f_{ij}q_i^{out} + \alpha_i q_i^{out} \tag{32}$$

$$\frac{\partial R_{ig}}{\partial q_{ig}} = \frac{1}{f_{ig}}\boldsymbol{v}_i'\boldsymbol{p}_i - \frac{q_{ig}}{f_{ig}^2}\boldsymbol{v}_i'\Lambda_i\boldsymbol{v}_i - \alpha_i\frac{q_{ig}}{f_{ik}^2} - \frac{1}{f_{ik}}\left[\Lambda_i\boldsymbol{v}_i\right]_g q_{ik} + p_g \tag{33}$$

$$= -\frac{1}{f_{ig}}\left[\Lambda_i\boldsymbol{v}_i\right]_k q_{ik} + p_g \tag{34}$$

So, the markup and markdowns are:

$$\begin{aligned}
\mu_i &= p_i^{out} - \frac{\partial C}{\partial q_i^{out}} \\
&= p_i^{out} - \sum_j p_j f_{ij} - \alpha_i q_i^{out} + \sum_j\left[\Lambda_i\boldsymbol{v}_i\right]_j f_{ij}q_i^{out} \\
&= \boldsymbol{v}_i'\boldsymbol{p}_i - \alpha_i q_i^{out} - \boldsymbol{v}_i'\Lambda_i\boldsymbol{v}_i q_i^{out} + \left[\Lambda_i\boldsymbol{v}_i\right]_i q_i^{out} \\
&= \left[\Lambda_i\boldsymbol{v}_i\right]_i q_i^{out} = \left[\Lambda_i\boldsymbol{q}_i\right]_i \\
\mu_{i,g} &= \left(-\frac{1}{f_{ik}}\left[\Lambda_i\boldsymbol{v}_i\right]_k q_{ik} + p_k\right) - p_k \\
&= -\left[\Lambda_i\boldsymbol{v}_i\right]_g q_i^{out}
\end{aligned}$$

So, the markup-markdown vector (with right signs) is: $\Lambda_i\boldsymbol{q}_i$. □

B.2 Proof of Proposition 1

The proof follows from the following three lemmas, proven in the Supplementary Appendix.

Lemma B.1.

$$(\overline{\Lambda}_i)^{-1} = (\Lambda_i^{out} + \Lambda_i^{in})^{-1} = \frac{\prod_{k\neq i} n_k B_k B_c}{\prod_{k\neq i} n_k B_k + B_c \sum_{j\neq i} \prod_{k\neq i, k\neq j} n_k B_k} \tag{35}$$

$$\overline{\Lambda}_N^{-1} = \frac{\prod_{k\neq 1} n_k B_k B_c}{\prod_{k\neq 1} n_k B_k + B_c \sum_{j\neq 1} \prod_{k\neq 1, k\neq j} n_k B_k} \tag{36}$$

Lemma B.2. Consider the equilibrium profile B^*. If $n_i > n_j$ then $\overline{BR}_i(X, B^*_{-i,j}) > \overline{BR}_j(X, B^*_{-i,j})$ for all $X \leq B_i^*, B_j^*$.

Lemma B.3. In equilibrium $n_i \geq n_j$ implies $B_i^* \geq B_j^*$.

We have that $\boldsymbol{v}_i = (1, -1)$ for each i. Moreover, by market clearing $q_i^{out} = q_j^{out} := Q$ for any i, j. So, using the expression for markups computed in REFERENCE, and noting that by Lemma B.3 if $n_i = n_j$ $B_i = B_j$ for all sectors and so market clearing conditions imply that $p_i - p_{i-1} = \frac{q}{B}$ and $\overline{\Lambda} = \overline{\Lambda}_i^{in} + \overline{\Lambda}_i^{out}$ are constant for any i, we get:

$$\mu_i^{out} = \frac{\Lambda_i^{out} \overline{\Lambda}^{-1}}{(1 + B(n-1)) + \overline{\Lambda}^{-1}} \frac{q}{B}$$

$$\mu_i^{in} = \frac{\Lambda_i^{in} \overline{\Lambda}^{-1}}{(1 + (n-1)B) + \overline{\Lambda}^{-1}} \frac{q}{B}$$

Now inspecting the right hand side of the expressions we see that the markup is increasing with Λ_i^{out}, which is itself increasing as one goes upstream. Then it follows that the markup is increasing going upstream, and symmetrically for the markdown. □

C Proofs of Section 5

C.1 Proof of Theorem 2

Part 1 By definition, the payoff is the objective function in REFERENCE. This function has Hessian matrix equal to $-\frac{1}{k_i} I_i - 2\Lambda_i$, and so is negative definite: so the payoff is strictly concave. Since the technology constraints are linear, we get that the first order conditions are sufficient and necessary for optimization, and the best reply equation REFERENCE still represents the equilibrium. The same equation shows that $\overline{B}_i \in [0, k_i]$, and Λ_i is continuous, so the best reply map is also continuous. So, by Brouwer's fixed point theorem, there exist an equilibrium.

The best reply equation and the assumption on Λ immediately allow to conclude that the best reply is increasing in the profile of slopes of other firms $\overline{B}_{-i}$. By Topkis' Theorem, the equilibrium set is a lattice, so it has a maximal and minimal element.

Part 2 Define BR^1, $BR^2 : \prod_i [0, k_i] \to \prod_i [0, k_i]$ the best reply maps for, respectively, model 1 and 2. We know that for any profile B we have, entrywise, $BR^1(B) > BR^2(B)$. Call $(B^*)^1$ the maximal equilibrium in model 1 and $(B^*)^2$ the maximal equilibrium in model 2. We have:

$$(B^*)^1 = BR^1((B^*)^1) > BR^2((B^*)^1)$$

Since the best reply is monotonic, we have that iterating the best reply of model 2 starting from $(B^*)^1$ we eventually reach the maximal equilibrium:

$$(B^*)^1 > BR^2((B^*)^1) > \cdots > (B^*)^2$$

which is what we wanted to show. The case of the minimal equilibrium works analogously.

□

C.2 Proof of Theorem 3

1. In the unilateral case the best reply equation works differently, because it is:

$$\max \boldsymbol{q}_i \boldsymbol{p}_i'$$

s.t. $\hat{\boldsymbol{q}}_i + (M - \hat{B}_i)\boldsymbol{p} = \boldsymbol{A}$. From this, we get the residual demand. But now, instead, the prices of inputs are fixed. Only the output price is allowed to change. For consistency, only the output quantity can affect it. We can express the equations more conveniently decomposing the matrix M as follows:

$$M = \begin{pmatrix} M_{i,down} & M_{i,down-up} \\ M'_{i,down-up} & M_{i,up} \end{pmatrix}$$

where, after reordering, $\mathcal{N}_{i,up}$ contains the subset of all goods that are inputs of i, or all goods that are directly or indirectly connected to inputs of i. $\mathcal{N}_{i,down}$ contains the remaining goods: the output of i, and possibly all downstream goods that are not connected to any inputs.

The equations involving $\boldsymbol{q}_i$ are:

$$\boldsymbol{q}_i + (M_{i,down} - \hat{B}_{i,down})\boldsymbol{p}_i^{down} + (M_{i,down,up} - \hat{B}_{i,down})\boldsymbol{p}_{i,in} = \boldsymbol{A}_i$$

We can solve only for the downstream prices:

$$\boldsymbol{p}_i^{down} = (M_{i,down} - \hat{B}_{i,down})^{-1} \left(\boldsymbol{A}_i - \boldsymbol{q}_i - ()\boldsymbol{p}_{i,up}\right)$$

and we get:

$$p_i^{out} = [(M_{i,down} - \hat{B}_{i,down})^{-1}]_{ii}(A_i - q_i^{out}) + const$$

So, the price impact is:

$$\Lambda_i^{unilateral} = \begin{pmatrix} [(M_{i,down} - \hat{B}_{i,down})^{-1}]_{ii} & 0 \\ 0 & 0 \end{pmatrix}$$

where $[(M_{i,down} - \hat{B}_{i,down})^{-1}]_{ii} = [(M_{ii} - B_{ii} - M'_{i,down} M^{-1}_{-i,down} M_{i,down})^{-1}]_{ii}$.

To compare with $\Lambda^{multilateral}$, we have to note that we can also decompose $\Lambda^{multilateral}$ as:

$$\Lambda^{multilateral} = \left[\left(\begin{pmatrix} M_{i,down} & M_{i,down-up} \\ M'_{i,down-up} & M_{i,up} \end{pmatrix} - \hat{B}_i \right)^{-1} \right]_{\mathcal{N}_i}$$

For simplicity, from now on denote the blocks of the matrix $M - \hat{B}_i$

as:

$$M - \hat{B}_i = \begin{pmatrix} A_1 & A_2 \\ A_2' & A_3 \end{pmatrix}$$

2. Using block inversion, $\Lambda_i^{unilateral}$ can also be written as:

$$\Lambda_i^{unilateral} = \lim_{T \to \infty} \left[\begin{pmatrix} A_1 & A_2 \\ A_2' & TA_3 \end{pmatrix}^{-1} \right]_{\mathcal{N}_i} \tag{37}$$

where I_i is the identity of appropriate dimension. But now, the matrix on the right-hand side is positive definite, so we get that a sufficient condition to conclude $\Lambda_i^{unilateral} \leq \Lambda_i^{multilateral}$ is:

$$\begin{pmatrix} A_1 & A_2 \\ A_2' & A_3 \end{pmatrix} - \begin{pmatrix} A_1 & A_2 \\ A_2' & TA_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & TA_3 \end{pmatrix} > 0$$

that is true, and so in the limit we obtain $\Lambda_i^{unilateral} \leq \Lambda_i^{multilateral}$.

3. For the case of the local market power it is immediate from the definition. If $\hat{\boldsymbol{q}}_i + (M - \hat{B}_i)\boldsymbol{p} = \boldsymbol{A}$, but the prices of the other markets are to be taken as given, then the equations involving $\boldsymbol{q}_i$ are:

$$\boldsymbol{q}_i + (M_i - B_i)\boldsymbol{p}_i + M_{i,-i}\boldsymbol{p}_{-i} = \boldsymbol{A}_i$$

and, inverting, we obtain:

$$\boldsymbol{p}_i = (M_i - B_i)^{-1} \left(\boldsymbol{A}_i - \boldsymbol{q}_i - M_{i,-i}\boldsymbol{p}_{-i} \right)$$

and the price impact is:

$$\Lambda_i^{local} = (M_i - B_i)^{-1},$$

so it is immediate to conclude:

$$\Lambda_i^{local} = (M_i - B_i)^{-1} < \left(M_i - B_i - (M_i^{out,in})' M_{-i}^{-1} M_i^{out,in} \right)^{-1} = \Lambda_i^{multilateral},$$

as we wanted to show.

C.3 Proof of Corollary 5.1

If there is a unique final good, say good 0, then the vector $\boldsymbol{A}$ has just one nonzero entry, corresponding to good 0. Moreover, the matrix $\hat{B}_c$ is composed by all zeros except a positive entry $B_{c,00}$ in the $0,0$ diagonal place. So, the vector $\boldsymbol{A} - \hat{B}_c \boldsymbol{f}_{0,L}$ is equal to: $(A_0 - B_{c,00}\hat{f}_{0,L})\boldsymbol{A}$, where $A_0 - B_{c,00}\hat{f}_{0,L} > 0$.

So, the price can be written:

$$p_0 = \frac{A_0 - B_{c,00}\hat{f}_{0,L}}{A_0}\boldsymbol{A}' M^{-1}\boldsymbol{A}$$

Now we know that M is increasing in each B_i, and so we obtain that p_0 is decreasing in each B_i.

C.4 Proof of Proposition 2

If the firms do not take the price impact into account on input markets, the best reply equations become:

$$B_i = \frac{\overline{\Lambda}_i^{-1} + (n-1)B_i}{\overline{\Lambda}_i^{-1} + (n-1)B_i + 1} \quad \text{where } \overline{\Lambda}_i = \frac{\Lambda_i^{out}}{1 + \Lambda_i^{out}}$$

and Λ_i^{out} is increasing upstream. Hence, in equilibrium, B_i is decreasing upstream, which means that markups are increasing.

Supplementary Appendix: TBA