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A Dynamic Stochastic Block Model for Multi-layer Networks

An application to international trade and public debt

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Motivation

Stylized Facts

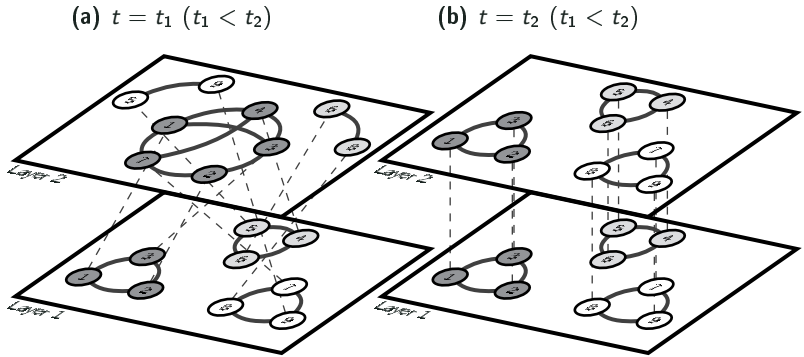
- **Economic blocks** formation, global value chains, technological differences and geography induce a significant level of clustering.
- The international relationships is a system with **multiple types of interactions**, e.g. trade flows of different commodities and debt trends.
- During global shocks, topological properties of the international relationships **have jointly changed**, i.e. composite risk.
- Higher-order effects due to a third party in a bilateral relationship.

Studies have explored the **community structure** in trade and public debt network, but separately and have not incorporated economic and statistical modeling.

Few statistical models deal with community detection in a **multi-layer graphs** context and they only focus on a correlational measure (clustering overlap) (e.g., Jovanovski and Kocarev, 2019; Stanley et al., 2016).

DSBM-M: Intuition

Figure 1: Dynamics of an undirected and unweighted network with two layers for the case of Layer 1 Granger-block-causing Layer 2. The plots show the network at period t_1 (Panel a) and t_2 with $t_1 < t_2$ (Panel b) and its block structure (nodes in a block have the same gray shade). Dashed lines indicate that all nodes appear in the two layers (multi-layer network).



Econometric methods

- An extension to existent **statistical models** for community detection (e.g., Matias and Miele, 2017 [JRSS]; Yang et al., 2011 [MachL])
 - A dynamic **multi-layer** model with **dependence** between the community structures (**Granger-block causality**).
- The model provide
 - A **partial pooling across edges and time** to capture some degree of heterogeneity but keeping efficiency.
 - A **unified framework** able to integrate different sources aspects of a **composite risk**: covariates, heterogeneous effects, community changes, and multi-layer dependence.
- Compared to the models in time series analysis using dependent Markov-chain (Agudze et al., 2021; Billio et al., 2016; Kaufmann, 2015)
 - It provides a more flexible specification that captures complex relationships.

Application to the international trade network and debt trend similarity

- We test for the **twin deficit** hypothesis from a network perspective, i.e. **interdependence** among countries.
- There is **no evidence** of Granger–block causality between trade flows (or growth) and debt similarity—no twin deficit phenomenon.
- The debt network presents **assortative group** and becomes a **core–periphery** structure during large global shocks.
- There is important evidence of **parameter heterogeneity**: in stable periods unemployment (real GDP per capita) differences lead to higher (lower) debt similarity, while income inequality and financial depth reduce debt similarity.
- Some EU countries are **not fully integrated** in the trade flow network, while the trade growth network present important differences.

Notation

A static multi-layer network is represented as an ordered triplet $\mathcal{G} = (\mathcal{V}, \mathcal{E}, Y)$.

- The set of **nodes** $\mathcal{V} = \{1, \dots, N\}$, e.g. countries.
- The set of **edges** $\mathcal{E} = \{\mathcal{E}^{(1)}, \dots, \mathcal{E}^{(L)}\}$, where $\mathcal{E}^{(\ell)} \subseteq \mathcal{V} \times \mathcal{V}$ includes the pairs actively interacting in layer ℓ , e.g. trade partnerships.
- The set $Y = \{Y^{(1)}, \dots, Y^{(L)}\}$, where square matrix $Y^{(\ell)}$ is the **adjacency matrix** for layer ℓ .

$Y^{(\ell)} \in \mathbb{R}^N \times \mathbb{R}^N$, where the (i, j) -th element of $Y^{(\ell)}$ is $Y_{ij}^{(\ell)} = 0$ if $(i, j) \notin \mathcal{E}^{(\ell)}$ and $Y_{ij}^{(\ell)} = a \in \mathbb{R} \setminus \{0\}$ if $(i, j) \in \mathcal{E}^{(\ell)}$.

Notice that if the network is undirected $Y^{(\ell)}$ is symmetric. If the network is unweighted $Y^{(\ell)}$ is a zero-one matrix.

In a **dynamic** framework, $\mathcal{G}_{1:T} = \{\mathcal{G}_t, t = 1, \dots, T\}$ with $\mathcal{V}_t = \mathcal{V}$.

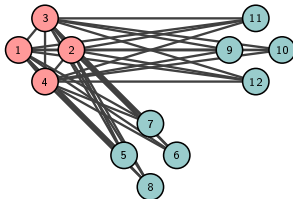
Community Detection

Figure 2: Examples of network structures

(a) Graph of blocks as groups



(b) Graph of blocks as core-periphery



- Communities refer to different network **structures of sub-graphs**.
- These structures are not easily reproduced by dyadic or tryadic relationships.

These structures are not easily reproduced by dyadic or tryadic relationships. Formally, \mathcal{V} is divided into a partition of Q elements, i.e. $\mathfrak{V}^{(\ell)} = \{\mathcal{V}_1^{(\ell)}, \dots, \mathcal{V}_Q^{(\ell)}\}$ with $\mathcal{V}_q^{(\ell)} \cap \mathcal{V}_r^{(\ell)} = \emptyset$ and $\cup_{q \in \mathcal{Q}^{(\ell)}} \mathcal{V}_q^{(\ell)} = \mathcal{V}$. SBM

DSBM-M: A probabilistic model

Considering a multi-layer dynamic network $\mathcal{G}_{1:T}$, the distribution for (i, j) in layer ℓ at t is

$$Y_{ijt}^{(\ell)} \mid X_{ijt}^{(\ell)}, Z_{it}^{(\ell)} = q, Z_{jt}^{(\ell)} = r, \vartheta^{(\ell)} \sim (1 - \nu_{qr}^{(\ell)})\delta(y) + \nu_{qr}^{(\ell)} f^{(\ell)}(y \mid \theta_{qr}^{(\ell)}, X_{ijt}^{(\ell)})$$

where

- The block membership of each node in layer ℓ evolves through time $\mathcal{T} = \{1, \dots, T\}$, following a Markov Chains $Z_{it}^{(\ell)} \in \mathcal{Q}^{(\ell)}$, with transition matrix $P_{it}^{(\ell)}$.
- The initial partition of Markov Chains is $\alpha^{(\ell)} = \{\alpha_1^{(\ell)}, \dots, \alpha_{Q^{(\ell)}}^{(\ell)}\}$ and $\nu_{qr}^{(\ell)}$ and $\theta_{qr}^{(\ell)}$ parameters related to sparsity of the network and weights.

DSBM-M: Layer dependence

The dependence between layers is driven by the transition probabilities,

$$\begin{aligned}\mathbb{P}(Z_{it}|Z_{it-1}) &= \mathbb{P}\left(Z_{it}^{(1)}|Z_{it-1}\right) \prod_{\ell=2}^L \mathbb{P}\left(Z_{it}^{(\ell)}|Z_{it}^{(1:\ell-1)}, Z_{it-1}\right) \\ &= \prod_{\ell=1}^L \mathbb{P}\left(Z_{it}^{(\ell)}|Z_{it-1}\right).\end{aligned}$$

In particular, a **saturated multinomial logit** specification (e.g., Everitt, 1992; Agresti, 2003),

$$\begin{aligned}\log\left(P_{it,q}^{(\ell)} C_{it}^{(\ell)}\right) &= \kappa_{0,q}^{(\ell)} + \underbrace{\sum_{m=1}^L W_{it-1}^{(m)'} \kappa_{m,q}^{(\ell)}}_{\text{main effects}} + \underbrace{\sum_{m>n} \left(W_{it-1}^{(m)'} \otimes W_{it-1}^{(n)'}\right) \kappa_{mn,q}^{(\ell)}}_{\text{first order effects}} \\ &\quad + \dots + \underbrace{\sum_{m=1}^L \left(W_{it-1}^{(m)'}\right) \kappa_{1\dots L,q}^{(\ell)}}_{(L-1)\text{-th order effects}}\end{aligned}$$

DSBM-M: Layer dependence

Let $Q^{(1)} = Q^{(2)} = 2$,

$$\begin{aligned}\log \left(P_{it,1}^{(1)} C_{it}^{(1)} \right) &= \kappa_{0,1}^{(1)} + \underbrace{\kappa_{\{1\},1,1}^{(1)} W_{it-1,1}^{(1)} + \kappa_{\{2\},1,1}^{(1)} W_{it-1,1}^{(2)}}_{\text{main effects}} + \underbrace{\kappa_{\{1,2\},1,1}^{(1)} W_{it-1,1}^{(1)} \cdot W_{it-1,1}^{(2)}}_{\text{first order effects}} \\ \log \left(P_{it,1}^{(2)} C_{it}^{(2)} \right) &= \kappa_{0,1}^{(2)} + \underbrace{\kappa_{\{1\},1,1}^{(2)} W_{it-1,1}^{(1)} + \kappa_{\{2\},1,1}^{(2)} W_{it-1,1}^{(2)}}_{\text{main effects}} + \underbrace{\kappa_{\{1,2\},1,1}^{(2)} W_{it-1,1}^{(1)} \cdot W_{it-1,1}^{(2)}}_{\text{first order effects}},\end{aligned}$$

the **Granger-block test** can be stated as:

- Debt trends does not Granger-block cause trade network

$$H_0 : \kappa_{\{2\},1,1}^{(1)} = \kappa_{\{1,2\},1,1}^{(1)} = 0$$

- Trade network does not Granger-block cause debt trends

$$H_0 : \kappa_{\{1\},1,1}^{(2)} = \kappa_{\{1,2\},1,1}^{(2)} = 0$$

DSBM-M application: Trade and debt trends

External and internal equilibrium is key in the analysis of public debt vulnerability and fiscal deficit

- Twin deficit phenomenon.
 - An increase in government spending can induce a higher level of imports depending on **external demand asymmetries** and openness (Rochon and Rossi, 2023).
 - International defaults influence reputation, reduce **trade credit**, capital flows, and may result in asset seizure (Rose, 2005).
 - An external deficit can impact the level output and public deficit (Nikiforos et al., 2015).
 - Internal trade frictions can increase the **cost of debt** (Feng et al., 2021).
- Economic policy trilemma.
 - Optimum currency area (Mundell, 1961): Under **asymmetries** a debt crisis may cover an external and banking crisis (Bibow, 2012).
 - Triffin dilemma (Triffin, 1978): Conflict between national and external monetary policy (e.g., US role in the Bretton Woods system).

Which deficit is creating vulnerabilities in the Euro area?

- Previous studies extracting a public debt network assume the **external deficit** play a limited role (e.g., García and Rambaud, 2023).
- VAR Granger–causality only consider **individual countries** or **bilateral relationships** (e.g., Nikiforos et al., 2015).

In order to address this question the **network structure of the currency area** should be taken into consideration.

The Granger–block causality studies the relationship at the **global topological properties** of the networks.

DSBM-M application: Data

We consider information for 27 European countries (EU) during the period 2003–2023:

- Monthly import values of Goods (CIF) in euros aggregated by quarter for all pair of countries $Y_{ijt}^{(1)}$. (Eurostat)
- Quarterly government debt in percentage of GDP. (Eurostat)
- Covariates related to Gravity Equation ($X^{(1)}$):
 - Quarterly GDPs, GDP_{it} , (Eurostat) and distance, d_{ij} , (Dynamic Database of USITC).
- Covariates related to debt ($X^{(2)}$):
 - Percentage of the population living in cities urb_{it} (annual), average annualized inflation by quarter $infl_{it}$ (monthly), average unemployment rate $unemp_{it}$ (monthly) by quarter, Gini coefficient $gini_{it}$ (annual), total financial sector liabilities in percentage of GDP $finc_{it}$ (annual), real GDP per capita $rgdpc_{it}$ (annual), real GDP growth $gdpg_{it}$ (quarterly). (Eurostat)

DSBM-M application: Data

The international trade flows are an **observed** network ($Y^{(1)}$).

The debt-to-GDP network ($Y_{ijt}^{(2)}$) is **extracted** using a rolling window **correlation** of the (first difference) debt-to-GDP for every pair of countries.

Date	$AD(S)$	$AID(S)$	De	APL^\dagger	WCC^\dagger	SCC^\dagger	$AClueCoe^\dagger$	$ABetCent^\dagger$
<i>Trade Flows Layer (weighted and directed): $Y_{ijt}^{(1)}$</i>								
2004 Q1	28.67	14.34	1	0	1	1	1	49.48
2008 Q4	39.41	19.7	1	0.01	1	1	1	40.7
2020 Q1	54.58	27.29	1	0.01	1	1	1	35.3
2023 Q1	77.1	38.55	1	0.01	1	1	1	39.41
<i>Debt Layer (unweighted and undirected): $\tilde{D}_{ijt}^{(2)} = \mathbb{I}_{\{(-0.5, 0.5)^\epsilon\}}(Y_{ijt}^{(2)})$</i>								
2004 Q1	7.04	7.04	0.27	2.04	2	2	0.6	12.56
2008 Q4	16.07	16.07	0.62	1.46	1	1	0.89	5.96
2020 Q1	13.41	13.41	0.52	1.5	1	1	0.74	6.56
2023 Q1	17.93	17.93	0.69	1.33	1	1	0.86	4.3

Specification 1

The first layer is the **trade flows** network,

$$Y_{ijt}^{(1)} \mid X_{ijt}, Z_{it}^{(1)} = q, Z_{jt}^{(1)} = r, \vartheta^{(1)} \sim \left(1 - \nu_{qr}^{(1)}\right) \delta(y) + \nu_{qr}^{(1)} \text{LN} \left(\left(\beta_{qr}^{(1)} \right)' X_{ijt}^{(1)}, \sigma_{qr}^{(1)2} \right), q, r \in \mathcal{Q}^{(1)}$$

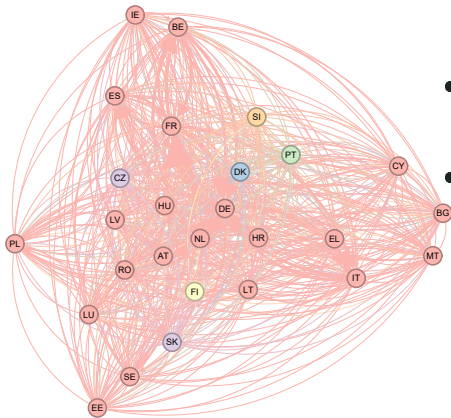
where $X_{ijt}^{(1)} = (1, \log GDP_{it}, \log GDP_{jt}, \log \text{dist}_{ij})'$.

The second layer is the debt similarity **dichotomized** in strong or weak correlation,

$$\tilde{D}_{ijt}^{(2)} \mid Z_{it}^{(2)} = q, Z_{jt}^{(2)} = r, \vartheta^{(2)} \sim \text{Bern} \left(\nu_{qr}^{(2)} \right), q, r \in \mathcal{Q}^{(2)}$$

DSBM-M application: Trade and debt trends

Figure 3: Trade flows network and communities (colors) in 2002Q4 under Specification 1



- The community structure of the trade flows is **highly persistent** and similar to a core–periphery.
- The core (1) presents a **higher variance**.
- The block analysis suggests EU is **highly integrated** to a lesser extend for: Finland, Czech Republic, Denmark, Slovenia, Slovakia, Hungary. [Details](#)

Specification 2

The first layer is the **trade growth** network,

$$\tilde{Y}_{ijt}^{(1)} \mid X_{ijt}, Z_{it}^{(1)} = q, Z_{jt}^{(1)} = r, \vartheta^{(1)} \sim \left(1 - \nu_{qr}^{(1)}\right) \delta(y) + \nu_{qr}^{(1)} \mathcal{N} \left(\left(\beta_{qr}^{(1)}\right)' \tilde{X}_{ijt}^{(1)}, \sigma_{qr}^{(1)2} \right), q, r \in \mathcal{Q}^{(1)}$$

where $\tilde{Y}_{ijt}^{(1)} = \Delta \log Y_{ijt}^{(1)}$ and similarly for $\tilde{X}_{ijt}^{(1)}$ (except the intercept and distance).

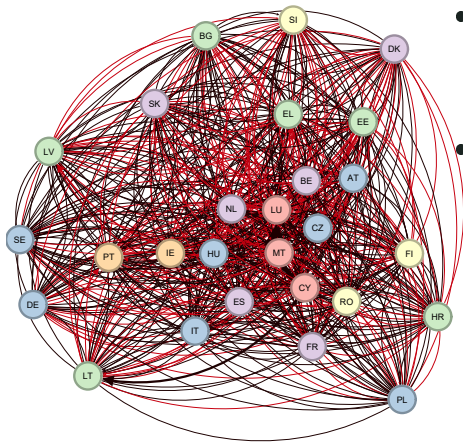
The second layer is the **debt correlation**,

$$\text{atanh}(Y_{ijt}^{(2)}) \mid Z_{it}^{(2)} = q, Z_{jt}^{(2)} = r, \vartheta^{(2)} \sim \left(1 - \nu_{qr}^{(2)}\right) \delta(y) + \nu_{qr}^{(2)} \mathcal{N} \left(\left(\beta_{qr}^{(2)}\right)' X_{ijt}^{(2)}, \sigma_{qr}^{(2)2} \right), q, r \in \mathcal{Q}^{(2)}$$

where $X_{ijt}^{(2)} = (1, |urb_{it} - urb_{jt}|, |infl_{it} - infl_{jt}|, |unemp_{it} - unemp_{jt}|, |gini_{it} - gini_{jt}|, |finc_{it} - finc_{jt}|, |rgdpc_{it} - rgdpc_{jt}|, |gdp_{it} - gdp_{jt}|)'$. 16/21

DSBM-M application: Trade and debt trends

Figure 4: Trade growth network and communities (node colors) in 2003Q1 under Specification 2

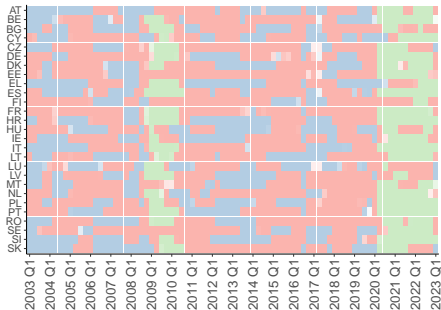


- The community structure of the trade growth is **more volatile and heterogeneous**.
- In particular community 2 and 4 seems to be more responsive to GDPs. In general community 2 (4) includes DE, AUS, PL, NL and IT (SE, FR and ES).

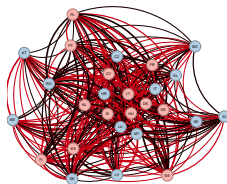
[Details](#)

DSBM-M application: Trade and debt trends

Figure 5: Dynamics of block memberships of countries (colors) in the debt network. Networks 2003Q1 and 2009Q1.



a) 2003Q1



b) 2009Q1

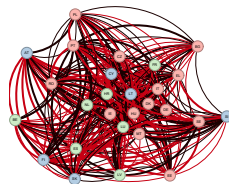


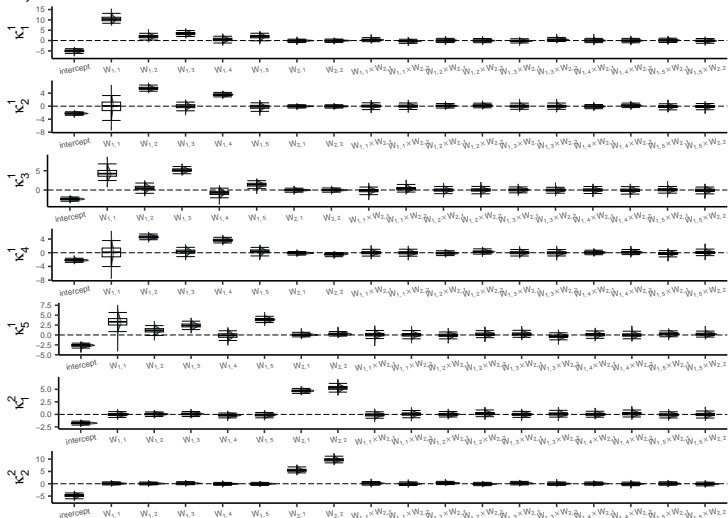
Table 1: Post. median and CI of $\beta_{0qr}^{(2)}$

$q \backslash r$	1	2	3
1	0.67 [0.65, 0.69]	-0.24 [-0.26, -0.21]	1.36 [1.32, 1.4]
2	-0.24 [-0.26, -0.21]	0.22 [0.18, 0.26]	0.51 [0.4, 0.62]
3	1.36 [1.32, 1.4]	0.51 [0.4, 0.62]	2.15 [2.11, 2.19]

Three communities are inferred from the debt correlation network: a (dis) assortative groups 1 and 2 towards (3) each other, while community 3 is a **core** active only during specific events: Great Recession and COVID-19/21

DSBM-M application: Trade and debt trends

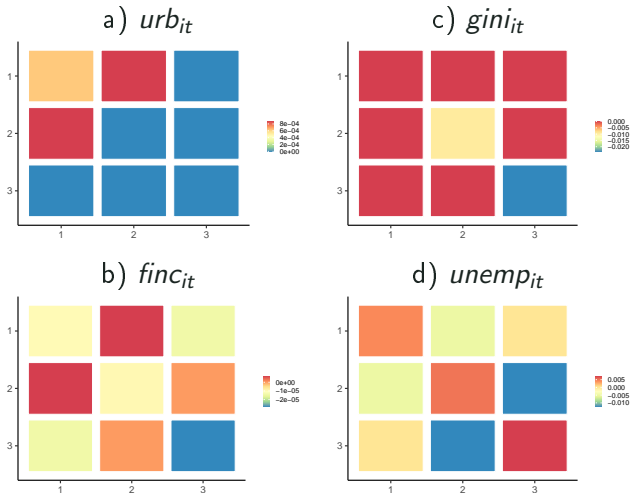
Figure 6: Transition probability parameters for the Trade (Layer 1) and debt (Layer 2) network



In general, there is **no block-causality** between trade and debt network. 19/21

DSBM-M application: Trade and debt trends

The **marginal effect** on debt similarity is highly **heterogeneous**: differences (similarity) in *urb* (*gini*) increase (reduce) debt correlation.



Conclusion

- DSBMM provides a **flexible** framework for community detection in a **multi-layer dynamic** setting with **dependence** beyond a correlational measure (Granger–block causality).
- We find **no evidence** of Granger–block causality between trade flows (or growth) and debt similarity in the period 2003Q1-2023Q1 for the EU members, i.e. no twin deficit phenomenon at community level.
- Trade flows network is a **persistent core–periphery** structure with few countries not integrated into the core, while the trade growth network is **continuously changing**.
- The debt network presents **assortative groups** and becomes a **core–periphery** structure during large global **shocks**.
- There is important evidence of parameter **heterogeneity**: in stable periods unemployment (real GDP per capita) differences lead to higher (lower) debt similarity, while income inequality and financial depth reduce debt similarity.

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Among the **statistical models** for community detection, SBM is a flexible framework. [Review](#)

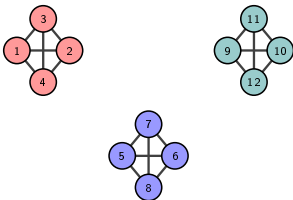
- SBM: Nodes are grouped into **communities** and the edge probabilities/weights depend on their membership.
- Therefore, in SBM the $Q \times Q$ **connectivity parameter** ν drives the edge formation process. For instance,

$$Y_{ij} | Z_i = q, Z_j = r \sim \text{Bern}(\nu_{qr})$$

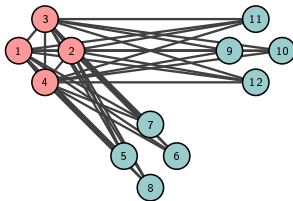
Notice: If $i \in \mathcal{V}_q$, then the node's membership is represented by the latent variable $Z_i = q$.

Figure 7: Network structures: An assortative behavior or blocks as groups (a,c) and a core-periphery structure(b,d).

(a) Graph of blocks as groups



(b) Graph of blocks as core-periphery



(c) Probability of an edge in blocks as groups

$$\nu = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) Probability of an edge in blocks as core-periphery

$$\nu = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

The complete likelihood is

$$L(Y, Z, D \mid X, \vartheta) = \prod_{\ell=1}^L \left\{ \prod_{t=1}^T \left(\prod_{i \neq j}^N \left[\left(1 - D_{ijt}^{(\ell)} \right) \delta \left(Y_{ijt}^{(\ell)} \right) + D_{ijt}^{(\ell)} f^{(\ell)} \left(Y_{ijt}^{(\ell)} \mid \theta_{Z_{it}^{(\ell)}}^{(\ell)} Z_{jt}^{(\ell)} \right) \right] \right. \right. \\ \left. \left. \left(\nu_{Z_{it}^{(\ell)}}^{(\ell)} Z_{jt}^{(\ell)} \right)^{D_{ijt}^{(\ell)}} \left(1 - \nu_{Z_{it}^{(\ell)}}^{(\ell)} Z_{jt}^{(\ell)} \right)^{1-D_{ijt}^{(\ell)}} \right) \right\} \\ \left(\prod_{t=2}^T \prod_{i=1}^N \mathbb{P}(Z_{it} \mid Z_{it-1}) \mathbb{P}(Z_{i1} \mid \alpha) \right)$$

- Markov chains $\{Z_i^{(\ell)}\}_{i=1}^N$ are conditional independents.
- The $Y_{1:T}^{(1:L)}$ components are independent conditional on $Z_{1:T}^{(1:L)}$ and $X_{1:T}^{(1:L)}$.
- By adding independent prior distributions a Bayesian inference can be performed through a full Gibbs-sampling.
- Time and layer dependence is driven only by the latent membership.

$$P_{it,q}^{(\ell)} = \frac{\exp \left(\widetilde{W}_{it-1} \kappa_q^{(\ell)} \right)}{\sum_{k=1}^{Q^{(\ell)}} \exp \left(\widetilde{W}_{it-1} \kappa_k^{(\ell)} \right)}$$

where $\widetilde{W}_{it-1} = \left(1, W_{it}^{(\ell)'} , \dots, \otimes_{m=1}^L \left(W_{it}^{(m)'} \right) \right)$ and $\kappa_q^{(\ell)} = \left(\kappa_{0,q}^{(\ell)}, \dots, \kappa_{1\dots L,q}^{(\ell)'} \right)$ a row and a column vector with $p+1$ elements, $p = \sum_{\mathcal{U} \subseteq 2^{\mathcal{L}}} s(\mathcal{U})$.

The Pólya–Gamma representation allows a full Gibbs and a more compact expression

$$h \left(\kappa^{(\ell)} \mid Z, \omega_{1:N1:T}^{(\ell)} \right) \propto \pi \left(\kappa^{(\ell)} \right) \exp \left(-\frac{1}{2} \left(\tilde{\xi}^{(\ell)} - \eta^{(\ell)} \right)' \Omega^{(\ell)} \left(\tilde{\xi}^{(\ell)} - \eta^{(\ell)} \right) \right)$$

$$\bar{\kappa}^{(\ell)} = \bar{K}^{(\ell)} \left(\left(I_{Q^{(\ell)}-1} \otimes \widetilde{W} \right)' \xi^{(\ell)} + \left(I_{Q^{(\ell)}-1} \otimes \widetilde{W} \right)' \Omega^{(\ell)} C^{(\ell)} + \left(\underline{K}^{(\ell)} \right)^{-1} \underline{\kappa}^{(\ell)} \right)$$

$$\bar{K}^{(\ell)} = \left(\left(I_{Q^{(\ell)}-1} \otimes \widetilde{W} \right)' \Omega^{(\ell)} \left(I_{Q^{(\ell)}-1} \otimes \widetilde{W} \right) + \left(\underline{K}^{(\ell)} \right)^{-1} \right)^{-1}$$

Table 2: Prior distributions for connectivity parameters and initial node partitions

Prior distribution	Family
$\beta_{qr}^{(\ell)} \sim \text{N} \left(\underline{\beta}_{qr}^{(\ell)}, \underline{\Sigma}_{qr}^{(\ell)} \right)$	Normal
$\sigma_{qr}^{(\ell)2} \sim \text{IG} \left(\underline{d}_{qr}^{(\ell)} / 2, \underline{e}_{qr}^{(\ell)} / 2 \right)$	Inverse-gamma
$\nu_{qr}^{(\ell)} \sim \text{Beta} \left(\underline{b}_{qr}^{(\ell)}, \underline{c}_{qr}^{(\ell)} \right)$	Beta
$\alpha^{(\ell)} \sim \text{Dir} \left(\underline{\alpha}^{(\ell)} \right)$	Dirichlet

The allocation variables $Z_i^{(\ell)}$ is sampled using forward-filter-backward sampling (FFBS) algorithm (Hamilton, 1994; Kim and Nelson, 1999)

Prediction Probability : $\mathbb{P} \left(Z_{it}^{(\ell)} = q \mid Z_{i,1:t-1}^{(-\ell)}, Z_{-i,1:t-1}^{(-\ell)}, \psi_{t-1}^{(\ell)} \right)$

Filtering Probability : $\mathbb{P} \left(Z_{it}^{(\ell)} = q \mid Z_{i,1:t}^{(-\ell)}, Z_{-i,1:t}^{(-\ell)}, \psi_t^{(\ell)} \right)$

Smoothing Probability : $\mathbb{P} \left(Z_{i,1:T}^{(\ell)} \mid Z_{i,1:T}^{(-\ell)}, Z_{-i,1:T}^{(-\ell)}, \psi_T^{(\ell)} \right)$

For the transition parameters, a **group-LASSO** is used to deal with the **overparameterization**, **collinearity** and **unbalanced** categories,

$$\min_{\kappa^{(\ell)}} -\ln h(Z^{(\ell)}|\kappa^{(\ell)}, Z^{(-\ell)}) + c^{(\ell)} \sum_{q \in \mathcal{Q}^{(\ell)} \setminus Q^{(\ell)}} \sum_{\mathcal{U} \in 2^{\mathcal{L}}} \|\kappa_{\mathcal{U},q}^{(\ell)}\|_2, \quad (1)$$

where $c^{(\ell)} > 0$ governs the sparsity level in the parameters $(\kappa^{(\ell)})$. The penalization in (1) has two components:

- 1 a l_1 -penalty that selects among the group of covariates;
- 2 and, a l_2 -penalty shrinking the parameters of highly correlated variables within each subset \mathcal{U} .

In a Bayesian framework, it is equivalent to **Multi-Laplacian prior**

$$\text{M-Laplace} \left(\kappa_{\mathcal{U},q}^{(\ell)} \mid 0, (c^{(\ell)})^{-1} \right) \propto (c^{(\ell)})^{s(\mathcal{U})} \exp \left(-c^{(\ell)} \left\| \kappa_{\mathcal{U},q}^{(\ell)} \right\|_2 \right),$$

where $c^{(\ell)} = [s(\mathcal{U})\rho^{(\ell)}]^{1/2}$ (Raman et al., 2009). If all covariate groups are singletons, i.e. $s(\mathcal{U}) = 1$ for all $\mathcal{U} \in 2^{\mathcal{L}}$, the group LASSO is equivalent to a **standard LASSO** (Park and Casella, 2008). It can be expressed as,

$$\begin{aligned} \kappa_q^{(\ell)} &\sim \text{N} \left(\underline{\kappa}_q^{(\ell)}, \underline{K}_q^{(\ell)} \right) \\ \underline{K}_q^{(\ell)} &= \text{diag} \left(\zeta_0^{(l)2}, \zeta_{q1}^{(l)2} 1'_{s(1)}, \dots, \zeta_0^{(l)2} 1'_{s(\ell)}, \dots, \zeta_{q\mathcal{U}}^{(l)2} 1'_{s(\mathcal{U})}, \dots, \zeta_{q1\dots L}^{(l)2} 1'_{s(1\dots L)} \right) \\ \zeta_{q\mathcal{U}}^{(l)2} | \rho^{(\ell)} &\sim \text{G} \left(\frac{s(\mathcal{U}) + 1}{2}, \frac{2}{\rho^{(l)} s(\mathcal{U})} \right) \\ \rho^{(l)} &\sim \text{G} \left(\iota_1^{(l)}, \iota_2^{(l)} \right) \end{aligned}$$

Table 3: List of countries and abbreviations

Austria (AT)	Spain (ES)	Latvia (LV)
Belgium (BE)	Finland (FI)	Malta (MT)
Bulgaria (BG)	France (FR)	Netherlands (NL)
Cyprus (CY)	Croatia (HR)	Poland (PL)
Czechia (CZ)	Hungary (HU)	Portugal (PT)
Germany (DE)	Ireland (IE)	Romania (RO)
Denmark (DK)	Italy (IT)	Sweden (SE)
Estonia (EE)	Lithuania (LT)	Slovenia (SI)
Greece (EL)	Luxembourg (LU)	Slovakia (SK)

Figure 8: Trade network community membership under Specification 1

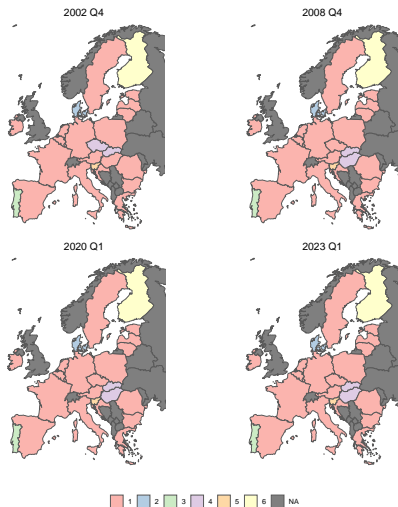


Figure 9: Marginal effect by pair of community interacting under Specification 1

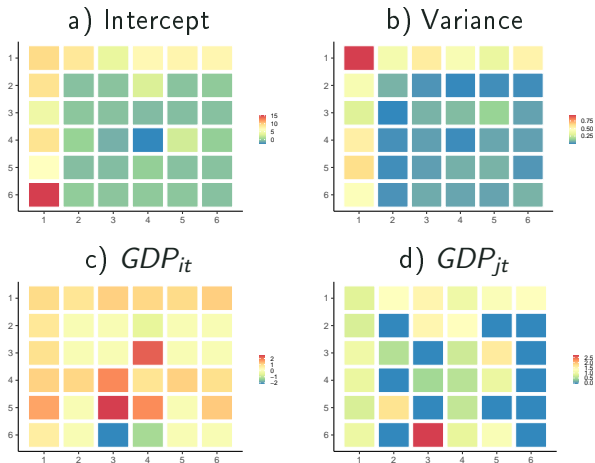


Figure 10: Dynamics of block memberships of countries in the Trade growth network under Specification 2.

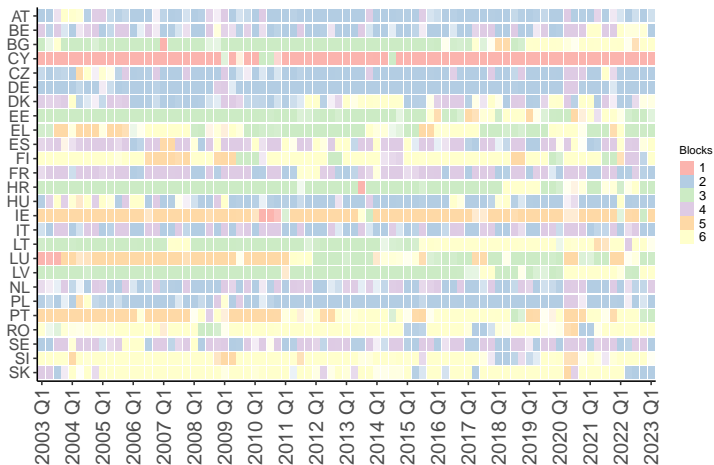
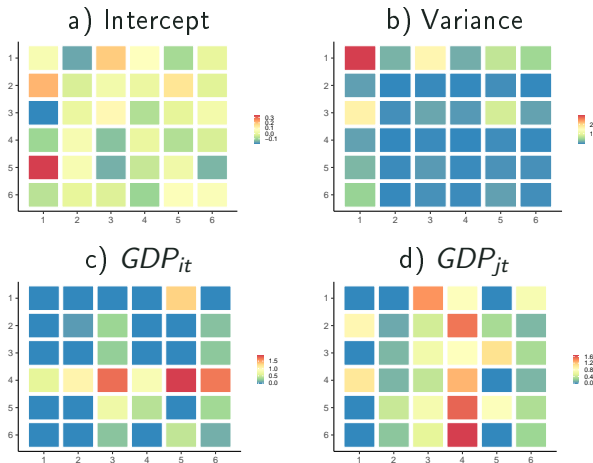


Figure 11: Marginal effect by pair of community interacting under Specification 2



Community detection methods can be based on **algorithms**, e.g. maximizing a modularity measure.

Using international trade network (ITN)

- For example, Barigozzi et al. (2011) and Bartesaghi et al. (2020) use numerical methods to identify communities of countries per product.
- Bartesaghi et al. (2019) propose an algorithm with a transformed adjacency matrix. Each entry refers to the similarity between nodes on different topological indicators (e.g. centrality).

One-layer network

- A Dynamic SBM (**DSBM**) is proposed by Yang et al. (2011) with (un)directed and unweighted networks. Matias and Miele (2017) extend it for time-varying parameters.
- SBM with intra-block degree heterogeneity are introduced by Karrer and Newman (2011); Zhao et al. (2012) and more recently Riverain et al. (2022) in a dynamic setting and Tan and Iorio (2019) assumes a static SBM with dynamic **node-time heterogeneity**.
- Other extensions deal with **mixed membership** and the **heterogeneity** of connectivity parameters across nodes (Airoldi et al., 2008; Zhao et al., 2012).

Multi-layer network

- Jovanovski and Kocarev (2019) identify communities per layer (local clustering), and a **consensus** of all layers (global clustering) with unweighted edges.